AN INTEGER VALUED HAUSDORFF-LIKE METRIC

BY

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*TO MY FAMILY
AND ESPECIALLY TO KARL*

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Abstract of Dissertation Presented to the Graduate Council
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Motivated by the need for a non-Euclidean metric be-
tween sets of objects and using the Hausdorff metric as a
prototype, an integer valued discrete analogue of the Haus-
dorff metric is presented. Basic properties common to both
metrics are examined, and a topological relationship is es-
tablished. The Hausdorff space of nonempty subsets of the
unit square is shown to be homeomorphic to the completion of
the direct limit of a sequence of finite spaces having the
discrete Hausdorff distance as a metric.

The potential usefulness of the discrete Hausdorff
metric in pattern recognition as applied to graded patterns,
or digitized grey level images, is examined and an algorithm
for computing the distance between two graded patterns is
presented. Also, a nonlinear neighborhood dependent

sharpening transformation particularly suited to graded
patterns is presented and an alternative proof of conver-
gence of the sharpening procedure is provided. Several
computations studying the interaction of the discrete Haus-
dorff metric with the sharpening transformation are reported.

INTRODUCTION

The evolution of the discipline of mathematical taxonomy [7] has increased the need for sensitive measures of similarity or distances between objects or classes of objects. The objects of interest may be biological sequences, digitized grey level images, individuals of a population, or elements of abstract point sets. Traditionally classification involved associating the objects to be classified with some n-dimensional vector in a Euclidean space, employing Euclidean distance as the measure of similarity and then clustering. The need for a non-Euclidean metric between sets of objects (clusters) was expressed in Jardine and Sibson [7].

The concept of distances between subsets of a metric space was first examined systematically by F. Hausdorff [6]. In particular, for nonempty closed subsets A and B of a compact metric space (X,d) the Hausdorff distance ρ_H between A and B is defined to be the maximum of the two values

$$\max_{x \in A} \min_{y \in B} d(x,y) \quad \text{and} \quad \max_{x \in B} \min_{y \in A} d(x,y).$$

Using the Hausdorff distance as a prototype, and motivated by the need for a non-Euclidean intertaxal distance, Bednarek and Smith [1] introduced an integer valued discrete analogue of the

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Hausdorff metric. The present work is devoted to a more detailed examination of this discrete Hausdorff metric and, in particular, to an examination of its potential for application in pattern recognition problems.

In Chapters I and II we discuss the original Hausdorff metric, the discrete analogue and their basic properties.

Chapter III contains one of the main results, namely, that the Hausdorff space of nonempty closed subsets of the unit square is the completion of the direct limit of a sequence of finite metric spaces having the discrete Hausdorff distance as a metric. This, in part, justifies the terminology and the use of the discrete metric for the study of digitized pictures.

A particularization, first suggested by Bednarek and Ulam [2], to the case of graded patterns (digitized grey level images) is examined more closely in Chapter IV. In particular, we provide an algorithm for the computation of the distance between two graded patterns and illustrate its application in the Appendix.

In [9], Kramer and Bruchner introduced a nonlinear neighborhood dependent sharpening transformation particularly applicable to the patterns considered in Chapter IV. In Chapter V we provide an alternate proof of the convergence of this sharpening procedure.

We feel that the relationships between our metric and the Kramer-Bruchner sharpening transformation, both being highly neighborhood dependent and both being potentially applicable to the processing and recognition of digitized images, were worth examining. Toward this end some computations were carried out and their results are reported in Chapter VI. While not definitive, these studies suggest some possible applications and additional investigations. Some of these are delineated in Chapter VII.

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CHAPTER I

BASIC DEFINITIONS AND ELEMENTARY
PROPERTIES OF THE HAUSDORFF METRIC

Let X be a set and let R^+ denote the nonnegative reals.

Definition 1.1. A function d from the cartesian product $X \times X$ into R^+ satisfying the following conditions for all elements x, y, and z of X is called a metric:

(1) d(x,y) = 0 implies that x = y

(2) d(x,x) = 0

(3) d(x,y) = d(y,x)

(4) d(x,z) ≤ d(x,y) + d(y,z).

The pair (X,d) is called a metric space.

Let (X,d) be a compact metric space, and let 2^X denote the family of all nonempty closed subsets of X. For elements A and B of 2^X, defining

$$\rho_H(A,B) = \max \{\max_{x \in A} \min_{y \in B} d(x,y), \max_{x \in B} \min_{y \in A} d(x,y)\}$$

we obtain the well-known metric on 2^X introduced by Hausdorff [6]. A topologically equivalent definition of the Hausdorff metric is given by $\rho_{H'}$, where

$$\rho_{H'}(A,B) = \max_{x \in A} \min_{y \in B} d(x,y) + \max_{x \in B} \min_{y \in A} d(x,y).$$

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When it is necessary to identify the metric d underlying the Hausdorff metric we will write ρ_{H_d} instead of ρ_H.

Let B be an element of 2^X, and x an element of X, where (X,d) is a compact metric space.

Definition 1.2. The distance from x to B, d(x,B) is given by d(x,B) = inf {d(x,b): b ∈ B}. For any positive number r, define U(B,r) = {x ∈ X: d(x,B) < r}.

If we let $N_r(b)$ denote the usual neighborhood about b of radius r, that is, $N_r(b)$ = {x ∈ X: d(x,b) < r}, then we observe that U(B,r) = $\cup${$N_r(b)$: b ∈ B}.

An equivalent and perhaps more intuitive definition of $\rho_H(A,B)$ is given by

$$\rho_H(A,B) = \inf \{r: A \subset U(B,r) \text{ and } B \subset U(A,r)\}.$$

Note that the value of the infimum is not always realized. That is, if $\rho_H(A,B) = r$, then it is possible that A is not a subset of U(B,r) or B is not a subset of U(A,r), as these are open sets. See Example 1.7 for an illustration of this situation.

Intuitively, the Hausdorff distance between sets A and B can be viewed as the smallest number ε such that an "ε-expansion" of the sets A and B will lead to mutual absorption, as shown in Figure I-1.
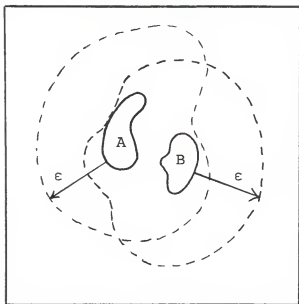
Figure I-1

Proposition 1.3. If A and B are elements of 2^X and $A \subset B$, then $U(A,r) \subset U(B,r)$.

Proof. This follows from the fact that if $A \subset B$, then inf $\{d(x,b): b \in B\} \leq$ inf $\{d(x,a): a \in A\}$.

Proposition 1.4. For A, B, C, and D elements of 2^X,
$$\rho_H(A \cup B, C \cup D) \leq \max \{\rho_H(A,C), \rho_H(A,D), \rho_H(B,C), \rho_H(B,D)\}.$$

Proof. Let $\rho_H(A,C) = q$, $\rho_H(A,D) = r$, $\rho_H(B,C) = s$, $\rho_H(B,D) = t$, and let $m = \max \{q,r,s,t\}$. Let ε be a positive number. By definition, $A \subset U(C, q + \varepsilon) \subset U(C, m + \varepsilon) \subset U(C \cup D, m + \varepsilon)$ and $B \subset U(C, s + \varepsilon) \subset U(C, m + \varepsilon) \subset U(C \cup D, m + \varepsilon)$, so that $A \cup B \subset U(C \cup D, m + \varepsilon)$. Analogously, $C \cup D \subset U(A \cup B, m + \varepsilon)$.

Then $\rho_H(A \cup B, C \cup D) \leq m + \varepsilon$ for every positive number ε, which implies that $\rho_H(A \cup B, C \cup D) \leq m$.

The following example shows that strict inequality is possible.

Example 1.5. Let $X = [0,1]$ and $d(x,y) = |x - y|$ for elements x and y of X. Let $A = [0,1/4]$, $B = [1/2,3/4]$, $C = [1/4,1/2]$, and $D = [3/4,1]$. Then $\rho_H(A,C) = 1/4$, $\rho_H(A,D) = 3/4$, $\rho_H(B,C) = 1/4$, $\rho_H(B,D) = 1/4$, and $\rho_H(A \cup B, C \cup D) = 1/4$ $< \max \{\rho_H(A,C), \rho_H(A,D), \rho_H(B,C), \rho_H(B,D)\} = 3/4$.

Corollary 1.6. For elements A and B of 2^X, $\rho_H(A \cup B, A)$ $\leq \rho_H(A,B)$.

Example 1.7. An example of strict inequality is given by letting $X = [0,1]$ with the usual metric; i.e., $d(x,y) = |x - y|$. Let $A = \{1/2\} \cup \{1/2 + (1/2)^k : k = 3,4,...\} = \{1/2,5/8,9/16,17/32,...\}$ and $B = \{1\}$. Then clearly $\rho_H(A,B) = 1/2$, while $\rho_H(A \cup B, A) = 3/8$ because $d(1,A) = \inf \{d(1,a): a \in A\} = \inf \{1/2,3/8,7/16,15/32,...\} = 3/8$.

Corollary 1.8. For elements A, B, and C of 2^X, $\rho_H(A \cup B, C) \leq \max \{\rho_H(A,C), \rho_H(B,C)\}$.

Proposition 1.9. For elements A, B, and C of 2^X, $\rho_H(A \cup B, C) \leq \rho_H(A,B) + \rho_H(B,C)$.

Proof. Let $\rho_H(A,B) = r$, $\rho_H(B,C) = s$, and $\varepsilon > 0$. Then

$A \subset U(B,r + \varepsilon)$, $B \subset U(C,s + \varepsilon) \subset U(C,r + s + 2\varepsilon)$, and
$C \subset U(B,s + \varepsilon) \subset U(A \cup B,r + s + 2\varepsilon)$. This implies that
for every $a \in A$ there exists a $b \in B$ such that $d(a,b)$
$< r + \varepsilon$, and for every $b \in B$ there exists a $c \in C$ such that
$d(b,c) < s + \varepsilon$. So $d(a,c) \leq d(a,b) + d(b,c) < r + s + 2\varepsilon$,
which implies that $A \cup B \subset U(C,r + s + 2\varepsilon)$. Thus,
$\rho_H(A \cup B,C) \leq r + s + 2\varepsilon$, but ε was arbitrary, so that
$\rho_H(A \cup B,C) \leq r + s$.

<u>Definition 1.10</u>. If X and Y are topological spaces, then a
continuous map f from X onto Y which is one-to-one and such
that f^{-1} is also continuous is called a <u>homeomorphism</u>.

<u>Definition 1.11</u>. Let (X_1,d_1) and (X_2,d_2) be metric spaces.
An <u>isometry</u> is a map f from X_1 onto X_2 such that $d_1(x,y) =$
$d_2(f(x),f(y))$ for all elements x and y of X_1.

Note that every isometry is a homeomorphism. In parti-
cular, if (X,d) is a compact metric space, then the map
f: $x \rightarrow \{x\}$ is an isometry of X onto a subspace of 2^X.

An interesting property of the Hausdorff metric, though
somewhat disappointing from a topological standpoint, is that
the topology of the underlying space (X,d) does not determine
the topology of $(2^X,\rho_{H_d})$. That is, two metrics d_1 and d_2
can generate the same topological spaces (X,d_1) and (X,d_2)
while the spaces $(2^X,\rho_{H_{d_1}})$ and $(2^X,\rho_{H_{d_2}})$ can be different.
For an example of this, see Kelley [8].

Borsuk [3] also points out that ρ_H does not measure the difference in topological structure of sets A and B. Thus, $\rho_H(A,B)$ can be arbitrarily small although the topological structures of A and B are quite different. For a broader discussion of the topological properties of 2^X, the reader is referred to [11].

CHAPTER II

A DISCRETE ANALOGUE

In this chapter we introduce an integer valued metric
which is analogous to the Hausdorff metric. Basic properties
and similarities between it and the Hausdorff metric are
examined. Several examples of the metric are included. This
metric was first introduced in a paper by Bednarek and
Smith [1], and the description which follows is found in [1].

Let X be a set with cardinality $|X|$ equal to n. For
most purposes, we only consider the case in which X is
finite. We assume that with every x in X there is associated
a unique nonempty subset N(x) of X called the neighborhood
of x. The only restriction that we place on N(x) is that
every point x is contained in its neighborhood, $x \in N(x)$.

For subsets A of X, define E(A) to be the set
$E(A) = \cup\{N(a): a \in A\}$. Recursively, define $E^2(A) = E(E(A))$,
and in general, $E^{k+1}(A) = E(E^k(A))$ for any positive integer
k. Define $E^0(A)$ to be just A itself, $E^0(A) = A$, and note
that $A = E^0(A) \subset E(A)$ by definition. Moreover,

$$E^0(A) \subset E^1(A) \subset E^2(A) \subset \ldots \subset E^k(A) \subset E^{k+1}(A) \subset \ldots .$$

We observe that the operator E is monotone, that is, if

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$A \subset B$ then $E(A) \subset E(B)$. We also note that E is _additive_, that is, $E(A \cup B) = E(A) \cup E(B)$.

We now define an integral metric on the nonempty subsets of X. Let A and B be two nonempty subsets of X and define the distance between A and B, $\rho(A,B)$, by

$$\rho(A,B) = \begin{cases} \min \{k: A \subset E^k(B) \text{ and } B \subset E^k(A)\} \\ |X| \text{ otherwise} \end{cases}$$

and note that $\rho(A,B) = |X|$ if and only if there is no positive integer k such that $A \subset E^k(B)$ and $B \subset E^k(A)$.

Theorem 2.1. If X is finite, then ρ is a metric on the nonempty subsets of X.

Proof. By definition $\rho(A,B)$ is a nonnegative integer. If $\rho(A,B) = 0$ then $A \subset E^0(B) = B$ and $B \subset E^0(A) = A$, which implies that $A = B$. Conversely, if $A = B$ then $A \subset B = E^0(B)$ and $B \subset A = E^0(A)$, so that $\rho(A,B) = 0$. The symmetry of ρ, $\rho(A,B) = \rho(B,A)$, follows directly from the definition. In order to prove the triangle inequality for ρ, let A, B, and C be nonempty subsets of X. We wish to show that $\rho(A,B) \leq \rho(A,C) + \rho(C,B)$. If either $\rho(A,C) = n$ or $\rho(C,B) = n$, where $|X| = n$, the result is true. Suppose that $\rho(A,C) = j$, $\rho(C,B) = k$, and that neither j nor k is equal to n. Then $A \subset E^j(C)$ and $C \subset E^k(B)$ so that $A \subset E^j(E^k(B)) = E^{j+k}(B)$. Also, $B \subset E^k(C)$ and $C \subset E^j(A)$ so that $B \subset E^k(E^j(A)) = E^{j+k}(A)$.

By definition of ρ this implies that $\rho(A,B) \le j + k = \rho(A,C) + \rho(C,B)$, and triangularity is established.

For the case in which X is infinite, we obtain an extended integer valued distance between sets A and B which fails to be a proper metric only in the sense that it can take on the value infinity.

<u>Corollary 2.2</u>. If X is infinite, ρ is an extended integer valued metric on the nonempty subsets of X.

We shall refer to this metric as either ρ or the discrete Hausdorff metric throughout further discussions. We now give several particular examples of the metric ρ.

<u>Example 2.3</u>. A <u>graph</u> G is a nonempty finite set of <u>points</u> (or <u>vertices</u>), V, together with a set E of unordered pairs of distinct points of V, called <u>edges</u>. We say two points v, w are <u>edged</u> if $(v,w) \in E$. We write G = (V,E).

Suppose G = (V,E) is a graph and let X = V. For points x in X, define the neighborhood of x, N(x), to be the set consisting of x and all those points edged with x; that is, $N(x) = \{x\} \cup \{y: (x,y) \in E\}$. Then ρ is a metric on graphs. In particular, consider the example given in Figure II-1. In Figure II-1, $B \subset E^3(A)$ and $A \subset E^4(B) = X$, so $\rho(A,B) = 4$.
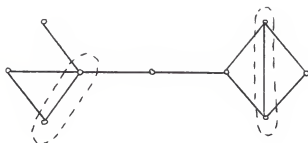
Figure II-1

<u>Example 2.4.</u> Let N_1 and N_2 be positive integers and let X
be the cells of an $N_1 \times N_2$ grid. Formally,
$X = \{(i,j): 1 \leq i \leq N_1, 1 \leq j \leq N_2\}$. For each cell x let
N(x) be a <u>cruciform neighborhood</u> consisting of those cells
to the left, right, above, and below x, as shown in
Figure II-2.

$$N(x) =$$
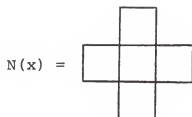
Figure II-2

N(x) can be defined formally by $N(x) = \{(h,k) \in X: |h - i|$
$+ |k - j| \leq 1\}$. An example of the expansion of a set A by
the operator E is shown in Figure II-3. The points of A
are indicated by * and the points of $E^k(A) - E^{k-1}(A)$ are
indicated by the number k for k = 1,2,3,...,7. Note that
$E^{14}(A) = X$ for all nonempty sets A of X if $N_1 = N_2 = 8$.

```
5 4 3 2 1 2 3 4
4 3 2 1 * 1 2 3
3 2 1 * * 1 2 3
3 2 1 * * 1 2 3
4 3 2 1 * * 1 2
5 4 3 2 1 1 2 3
6 5 4 3 2 2 3 4
7 6 5 4 3 3 4 5
```

Figure II-3

Many properties of the Hausdorff metric also hold for the discrete Hausdorff metric. The following sequel is a partial list of these analogous properties. Because the proofs for the case in which X is infinite are the same as those for the finite case, we shall only consider sets X with cardinality $|X| = n$ where n is finite.

Proposition 2.5. Let A, B, C, and D be nonempty subsets of X. Then $\rho(A \cup B, C \cup D) \leq \max \{\rho(A,C), \rho(A,D), \rho(B,C), \rho(B,D)\}$.

Proof. Let $\rho(A,C) = h$, $\rho(A,D) = i$, $\rho(B,C) = j$, and $\rho(B,D) = k$. If h, i, j, or k equals n, then the result follows as $\rho(A \cup B, C \cup D) \leq n$ by definition. Suppose h, i, j, and k are all less than n. Let $m = \max \{h,i,j,k\}$. Then $A \subset E^h(C) \subset E^m(C) \subset E^m(C \cup D)$ and $B \subset E^j(C) \subset E^m(C) \subset E^m(C \cup D)$, so that $A \cup B \subset E^m(C \cup D)$. Similarly, $C \subset E^m(A) \subset E^m(A \cup B)$ and $D \subset E^m(A) \subset E^m(A \cup B)$, so $C \cup D \subset E^m(A \cup B)$. Thus $\rho(A \cup B, C \cup D) \leq m$.

By letting C = D = A, we obtain the following corollary.

Corollary 2.6. Let A and B be nonempty subsets of X. Then
$\rho(A \cup B, A) \leq \rho(A, B)$.

Example 2.7. A simple example showing that strict inequality
can hold in Corollary 2.6 is shown in Figure II-4, where
the cells of set A are indicated by a and the cells of B by
b. Cruciform neighborhoods (see Example 2.4) are used in
this example to compute $\rho(A, B)$. In this example,
$\rho(A \cup B, A) = 1 < 4 = \rho(A, B)$.

b	a		a
	a	a	a

Figure II-4

Proposition 2.8. Let A, B, and C be nonempty subsets of X.
Then $\rho(A \cup B, C) \leq \rho(A, B) + \rho(B, C)$.

Proof. Let $\rho(A, B) = j$ and $\rho(B, C) = k$. If either $j = n$ or
$k = n$, then the result follows, so assume j and k are not
equal to n. Then $A \subset E^j(B)$ and $B \subset E^k(C)$ so that
$A \cup B \subset E^j(B) \subset E^j(E^k(C)) = E^{j+k}(C)$. Also, $C \subset E^k(B)$ and
$B \subset E^j(A)$ so $C \subset E^k(E^j(A)) = E^{j+k}(A) \subset E^{j+k}(A \cup B)$.
Thus $\rho(A \cup B, C) \leq j + k$.

At this point it would seem that increasing the cardinality of sets decreases the resulting metric value. This is shown to be false in Figure II-5 where $A \subset B$ but $\rho(B,C) > \rho(A,C)$. The cells of A are indicated by *, the cells of B by □, and the cells of C by #. In this example $\rho(B,C) = 3$ while $\rho(A,C) = 1$, again using cruciform neighborhoods.

Figure II-5

The next proposition is an observation of how the choice of neighborhoods affects the value of the resulting metrics. Suppose that for every point x in X there are associated two neighborhoods $N_1(x)$ and $N_2(x)$. Associated with these neighborhoods are two metrics ρ_1 and ρ_2 respectively.

<u>Proposition 2.9</u>. If $N_1(x) \subset N_2(x)$ for every $x \in X$, then $\rho_2(A,B) \leq \rho_1(A,B)$ for all nonempty subsets A and B of X.

<u>Proof</u>. Let $\rho_1(A,B) = j$, and note that if $j = n$ the result is immediate, so assume that $j < n$. Let $E_i(A) = \cup\{N_i(a): a \in A\}$ for $i = 1,2$, and in general, $E_i^k(A) = E_i(E_i^{k-1}(A))$ for $i = 1,2$. Then $E_1^k(A) \subset E_2^k(A)$ for all

nonnegative integers k and any nonempty set A since $N_1(a)$ is a subset of $N_2(a)$ for all a ϵ A. This implies that $A \subset E_1^j(B) \subset E_2^j(B)$, and conversely that $B \subset E_1^j(A) \subset E_2^j(A)$, so that $\rho_2(A,B) \leq j = \rho_1(A,B)$.

CHAPTER III

TOPOLOGICAL CONNECTION

The definition of the metric ρ suggests a strong connection between it and the Hausdorff metric. In order to relate the two it is necessary to decide the neighborhoods of points in the discrete case and the underlying metric in the Hausdorff case. This chapter will establish a topological relationship between the unit square with the Minkowski metric and an appropriate choice of neighborhoods for the discrete version.

Let I denote the unit interval and consider the metric space (I^2, d) where $d((x_1, x_2), (y_1, y_2)) = \max\{|x_i - y_i|: i = 1, 2\}$. Let $\underline{A}$ denote the family of all nonempty closed subsets of I^2. Then the Hausdorff metric ρ_H defines a distance between elements of $\underline{A}$.

For all positive integers n, let X_n be the set of all ordered pairs of binary sequences of length n; that is, $X_n = \{(a_1 \ldots a_n, b_1 \ldots b_n): a_i, b_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, n\}$. For simplicity of notation, denote elements of X_n $(a_1 \ldots a_n, b_1 \ldots b_n)$ by $(a_i, b_i)_n$. Let A_n be the collection of all nonempty subsets of X_n for all positive integers n.

For points $(a_i, b_i)_n$ of X_n, define the neighborhood to be $N((a_i, b_i)_n) = \{(u_i, v_i)_n \in X_n : |a_1 \cdots a_{n(2)} - u_1 \cdots u_{n(2)}| \leq 1$ and $|b_1 \cdots b_{n(2)} - v_1 \cdots v_{n(2)}| \leq 1\}$, where $a_1 \cdots a_{n(2)}$ denotes the integer equivalent to the base 2 number $a_1 \cdots a_n$. Let ρ_n be the discrete Hausdorff metric defined on $A_n \times A_n$ with the above definition of neighborhoods. We now have a sequence of metric spaces (A_n, ρ_n) for $n \geq 1$.

Observe that the binary sequences uniquely determine numbers L and M such that $a_1 \cdots a_{n(2)} = L$, $b_1 \cdots b_{n(2)} = M$, and $0 \leq L, M \leq 2^n - 1$. Thus, there is a natural one-to-one correspondence between nonempty subsets of X_n and certain closed subsets of the unit square given by

$$\Phi_n : A_n \rightarrow \underline{A}:$$
$$\{(a_i, b_i)_n\} \rightarrow [L/2^n, (L+1)/2^n] \times [M/2^n, (M+1)/2^n].$$

Define Φ_n of a general element of A_n by distributing over unions in the obvious manner. Consequently, there is a one-to-one correspondence between the elements of A_n and certain closed subsets of I^2. Applying Φ_n to neighborhoods of points in X_n gives rise to the following correspondence:

$$\Phi_n(N((a_i, b_i)_n)) = \{(x,y) \in I^2 : (L-1)/2^n \leq x \leq (L+2)/2^n,$$
$$(M-1)/2^n \leq y \leq (M+2)/2^n\}.$$

If $(a_i, b_i)_n$ is an element of X_n, denote $E^k(\{(a_i, b_i)_n\})$ by $E_n^k(a_i, b_i)_n$ for all nonnegative integers k. Then

$\Phi_n(E_n^k(a_i,b_i)_n) = \{(x,y) \in I^2: (L-k)/2^n \le x \le (L+k+1)/2^n,$
$(M-k)/2^n \le y \le (M+k+1)/2^n\}.$

Now let A be an element of $\underline{A}$ and define the n-th
encoding of A, $e_n(A)$, by

$e_n(A) = \cup\{(a_i,b_i)_n \in X_n: A \cap \Phi_n((a_i,b_i)_n)$ is nonempty$\}.$

Note that $e_n(A)$ is an element of A_n. Denote $\Phi_n(e_n(A))$ by
$\Phi_n e_n A$.

The n-th encoding of a set A can be interpreted as
placing a $2^n \times 2^n$ grid over the unit square, and the cells
of the grid which intersect A are the encoding. An illustra-
tion of this situation is given in Figure III-1.

One of the main results of this section is that for
elements A and B of $\underline{A}$, $\rho_H(A,B) = \lim_{n \to \infty} \dfrac{\rho_n(e_n(A),e_n(B))}{2^n}$.
The following lemmas will be used to establish this result.

Lemma 3.1. $\rho_H(A,\Phi_n e_n A) \le 1/2^n$.

Proof. By the definition of $e_n(A)$, A is a subset of $\Phi_n e_n A$.
Let x be an element of $\Phi_n e_n A$. By the one-to-one correspon-
dence and the definition of $e_n(A)$, there is an element a of
A such that $d(a,x) \le 1/2^n$. It follows that $d(x,A) \le 1/2^n$
and the result follows.
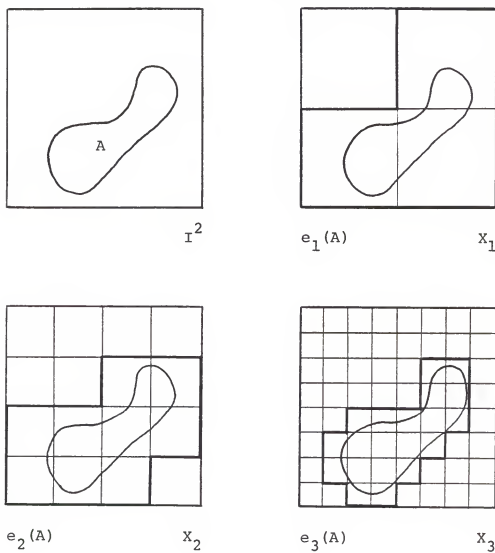
Figure III-1

<u>Lemma 3.2.</u> For elements A and B of $\underline{A}$, if $\rho_H(\Phi_n e_n A, \Phi_n e_n B)$ $\leq k/2^n$, then $\rho_n(e_n(A), e_n(B)) \leq k$.

<u>Proof.</u> $\rho_H(\Phi_n e_n A, \Phi_n e_n B) \leq k/2^n$ if and only if for every element a of $\Phi_n e_n A$ there is a b' in $\Phi_n e_n B$ so that $d(a,b')$ $\leq k/2^n$, and for every b in $\Phi_n e_n B$ there is an a' in $\Phi_n e_n A$ so that $d(a',b) \leq k/2^n$. Let $a = (\alpha_1, \alpha_2)$ be an element of $\Phi_n e_n A$. Let $b = (\beta_1, \beta_2)$ be an element of $\Phi_n e_n B$ such that $d(a,b) \leq k/2^n$. Then by the one-to-one correspondence, there are points $(a_i, b_i)_n$ in $e_n(A)$ and $(q_i, r_i)_n$ in $e_n(B)$ so that the following is true:

$$a_1 \cdots a_{n(2)} = L, \; b_1 \cdots b_{n(2)} = M,$$
$$q_1 \cdots q_{n(2)} = Q, \; r_1 \cdots r_{n(2)} = R,$$
$$L/2^n \leq \alpha_1 \leq (L+1)/2^n, \; M/2^n \leq \alpha_2 \leq (M+1)/2^n,$$
$$Q/2^n \leq \beta_1 \leq (Q+1)/2^n, \; R/2^n \leq \beta_2 \leq (R+1)/2^n.$$

Then $d(a,b) \leq k/2^n$ if and only if $|\alpha_i - \beta_i| \leq k/2^n$ for $i = 1,2$. So $(Q-k)/2^n \leq \beta_1 - k/2^n \leq \alpha_1 \leq \beta_1 + k/2^n \leq$ $(Q+k+1)/2^n$ and $(R-k)/2^n \leq \beta_2 - k/2^n \leq \alpha_2 \leq \beta_2 + k/2^n \leq$ $(R+k+1)/2^n$. This implies that a is an element of $\Phi_n(E_n^k(q_i, r_i)_n)$. By the one-to-one correspondence, this means that $e_n(A)$ is a subset of $E_n^k(e_n(B))$. By a symmetric argument, $e_n(B)$ is a subset of $E_n^k(e_n(A))$. The result follows from the definition of ρ_n.

<u>Lemma 3.3.</u> For elements A and B of $\underline{A}$, if $\rho_H(\Phi_n e_n A, \Phi_n e_n B)$ $> k/2^n$, then $\rho_n(e_n(A), e_n(B)) \geq k$.

<u>Proof</u>. If $\rho_H(\Phi_n e_n A, \Phi_n e_n B) > k/2^n$, we can assume without loss of generality that there is a point $a = (\alpha_1, \alpha_2)$ in $\Phi_n e_n A$ so that for every point $b = (\beta_1, \beta_2)$ in $\Phi_n e_n B$, $d(a,b) > k/2^n$. Corresponding to b is the point $(q_i, r_i)_n$ in X_n where $\Phi_n((q_i, r_i)_n) = [Q/2^n, (Q+1)/2^n] \times [R/2^n, (R+1)/2^n]$. If $d(a,b) > k/2^n$, then $|\alpha_i - \beta_i| > k/2^n$ for some $i = 1,2$. Assume without loss of generality that $|\alpha_1 - \beta_1| > k/2^n$. Then either $\alpha_1 > \beta_1 + k/2^n$ or $\alpha_1 < \beta_1 - k/2^n$. But $Q/2^n \leq \beta_1 \leq (Q+1)/2^n$ implies that either $\alpha_1 > (Q+k)/2^n$ or $\alpha_1 < (Q+1-k)/2^n$. Then a is not an element of $\{(x,y) \in I^2 : (Q+1-k)/2^n \leq x \leq (Q+k)/2^n, (R+1-k)/2^n \leq y \leq (R+k)/2^n\} = \Phi_n(E_n^{k-1}(q_i, r_i)_n)$. This implies that $e_n(A)$ is not a subset of $E_n^{k-1}(e_n(B))$. So if $\rho_H(\Phi_n e_n A, \Phi_n e_n B) > k/2^n$, then $\rho_n(e_n(A), e_n(B)) \geq k$.

The main result of this section now follows from the lemmas.

<u>Theorem 3.4</u>. For elements A and B of $\underline{A}$,

$$\rho_H(A,B) = \lim_{n \to \infty} \frac{\rho_n(e_n(A), e_n(B))}{2^n}.$$

<u>Proof</u>. Let $\rho_H(A,B) = r$. If $r = 0$ then $A = B$ and $e_n(A) = e_n(B)$ for all $n = 1, 2, \ldots$, and the result is immediate. Assume that r is positive. Then because the dyadic rationals are dense in $[0,1]$, there is a sequence $\{k_n/2^n : k_n$ is a positive integer, $n \geq 1\}$, so that $(k_n - 1)/2^n < r \leq k_n/2^n$ for $n = 1, 2, \ldots$, and $\lim_{n \to \infty} k_n/2^n = r$.

.

By the triangle inequality and Lemma 3.1,

$\rho_H(\phi_n e_n A, \phi_n e_n B) \leq \rho_H(\phi_n e_n A, A) + \rho_H(A, B) + \rho_H(B, \phi_n e_n B) \leq$
$(k_n + 2)/2^n$. Also, $\rho_H(\phi_n e_n A, \phi_n e_n B) \geq \rho_H(A, B) - \rho_H(A, \phi_n e_n A)$
$- \rho_H(B, \phi_n e_n B) > (k_n - 3)/2^n$. By Lemma 3.2 and Lemma 3.3, this
implies that $k_n - 3 \leq \rho_n(e_n(A), e_n(B)) \leq k_n + 2$. So
$(k_n - 3)/2^n \leq (\rho_n(e_n(A), e_n(B)))/2^n \leq (k_n + 2)/2^n$. By taking the
limit as n tends to infinity, the result follows.

Thus we see that the Hausdorff distance between two
nonempty closed subsets of I^2 can be approximated by using
the metric ρ on the encodings of the two sets. This
suggests that a stronger connection exists between the Haus-
dorff space $(\underline{A}, \rho_H)$ and the discrete metric ρ. In order to
establish the topological connection, we must consider the
direct limit of the spaces A_n. The following definitions
are found in Dugundji [5].

Definition 3.5. A binary relation < on a set A is called a
preorder if it is reflexive and transitive, that is,

    (1) a < a for all a in A

    (2) a < b and b < c implies that a < c for all a, b,
        and c in A.

A set together with a definite preorder is called a
preordered set. A preordered set A with the additional pro-
perty that for all a and b in A there is a point c in A such
that a < c and b < c is called a directed set.

Definition 3.6. Let D be a directed set, and let
$\{X_\alpha : \alpha \in D\}$ be a family of spaces indexed by D. If for
every pair of indices α and β with $\alpha < \beta$ there exists a
continuous map $\theta_{\alpha,\beta}: X_\alpha \to X_\beta$ such that whenever $\alpha < \beta < \gamma$,
then $\theta_{\alpha,\gamma} = \theta_{\beta,\gamma} \circ \theta_{\alpha,\beta}$, then the family of spaces and maps
$\{X_\alpha, \theta_{\alpha,\beta}\}$ is called a __direct spectrum over D__. The maps $\theta_{\alpha,\beta}$
are called __connecting maps__, and the image of $x_\alpha \in X_\alpha$ under
any connecting map is called a __successor__ of x_α.

Definition 3.7. Let $\{X_\alpha, \theta_{\alpha,\beta}\}$ be a direct spectrum. Let R
be the equivalence relation given by

$x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$, $x_\alpha R x_\beta$ if and only if x_α and x_β have
a common successor.

Then the quotient space $\sum_\alpha X_\alpha / R$ is called the __direct limit of__
__the spectrum__, and is denoted by X^∞.

We now define a direct limit of the spaces A_n. For all
positive integers n, define a connecting map $\theta_{n,n+1}: A_n \to A_{n+1}$
by $\theta_{n,n+1}(\{(a_i,b_i)_n\}) = \cup\{(u_i,v_i)_{n+1} \in X_{n+1}: \phi_n(\{(a_i,b_i)_n\})$
$= \phi_{n+1}(\cup\{(u_i,v_i)_{n+1}\})\}$. For an arbitrary element of A_n,
extend this map over unions of points in X_n by distributing
$\theta_{n,n+1}$ over unions; i.e., $\phi_n = \phi_{n+1} \circ \theta_{n,n+1}$ for all elements
of A_n. For integers j and k with $j < k$, define $\theta_{j,k}: A_j \to A_k$
to be the composition of the connecting maps:

$$\theta_{j,k} = \theta_{k-1,k} \circ \theta_{k-2,k-1} \circ \cdots \circ \theta_{j,j+1}.$$

If $j \leq k \leq 1$, then $\theta_{j,1} = \theta_{k,1} \circ \theta_{j,k}$, and because the spaces A_n are discrete, the maps $\theta_{j,k}$ are continuous. Thus, $\{A_n, \theta_{n,m}\}$ forms a direct spectrum, and we can consider the direct limit A^∞ of the spaces $\{A_n\}$. The elements of A^∞ are equivalence classes of elements of $\sum_n A_n$ under the relation R where xRy if and only if $\theta_{j,k}(x) = \theta_{r,k}(y)$ for some k, where $x \in A_j$, $y \in A_r$, and both j and r are less than or equal to k. Denote the elements of A^∞ by $[x]$.

<u>Lemma 3.8.</u> $[x]$ is an element of A^∞ if and only if $[x] = [e_n(A)]$ for some positive integer n and some $A \in \underline{A}$.

<u>Proof.</u> If $[x]$ is an element of A , then $[x]$ is the equivalence class of some element a_n of A_n, $[x] = [a_n]$. Then

$$\Phi_n(a_n) = \cup_{i=1}^{k} \{[L_i/2^n, (L_i+1)/2^n] \times [M_i/2^n, (M_i+1)/2^n]\},$$

where $0 \leq L_i, M_i \leq 2^n - 1$ for $i = 1, 2, \ldots, k$.

Let $A = \cup_{i=1}^{k} \{((2L_i+1)/2^{n+1}, (2M_i+1)/2^{n+1})\}$. Then A is an element of $\underline{A}$ and $e_n(A) = a_n$, which implies that $[e_n(A)] = [a_n] = [x]$. The opposite implication follows from the definition of A as the quotient space $\sum_n A_n/R$, and the fact that $e_n(A)$ is an element of A_n.

<u>Lemma 3.9.</u> For all positive integers j and k, and elements A and B of $\underline{A}$,

$$\rho_j(e_j(A), e_j(B)) = \frac{\rho_{j+k}(\theta_{j,j+k}(e_j(A)), \theta_{j,j+k}(e_j(B)))}{2^k}.$$

Proof. The proof follows by induction on k. Consider the case when k = 1. Let $\rho_j(e_j(A), e_j(B)) = m$. If m = 0 then $e_j(A) = e_j(B)$ and the result follows. Assume m is positive, then $e_j(A) \subset E_j^m(e_j(B))$, $e_j(B) \subset E_j^m(e_j(A))$, and we can assume without loss of generality that $e_j(A)$ is not a subset of $E_j^{m-1}(e_j(B))$. Then there exists an element $(a_i, b_i)_j$ of $e_j(A)$ such that $(a_i, b_i)_j \in E_j^m(q_i, r_i)_j$ for some $(q_i, r_i)_j \in e_j(B)$, but $(a_i, b_i)_j$ is not an element of $E_j^{m-1}(u_i, v_i)_j$ for all elements $(u_i, v_i)_j$ of $e_j(B)$. Then

$$[L/2^j, (L+1)/2^j] \times [M/2^j, (M+1)/2^j] =$$

$$\Phi_j(\{(a_i, b_i)_j\}) \subset \Phi_j(E_j^m(q_i, r_i)_j) =$$

$$[(Q-m)/2^j, (Q+m+1)/2^j] \times [(R-m)/2^j, (R+m+1)/2^j].$$

Then by the one-to-one correspondence and the equation $\Phi_n = \Phi_{n+1} \circ \theta_{n,n+1}$, we have the following:

$$\theta_{j,j+1}(e_j(A)) \subset E_{j+1}^{2m}(\theta_{j,j+1}(e_j(B))) \text{ and}$$

$$\theta_{j,j+1}(e_j(B)) \subset E_{j+1}^{2m}(\theta_{j,j+1}(e_j(A))).$$

This implies that $\rho_j(\theta_{j,j+1}(e_j(A)), \theta_{j,j+1}(e_j(B))) \leq 2m$. If

$$\Phi_j(\{(a_i, b_i)_j\}) = \Phi_{j+1}(\theta_{j,j+1}(\{(a_i, b_i)_j\})) \subset$$

$$\Phi_{j+1}(E_{j+1}^{2m-1}(\theta_{j,j+1}(\{(u_i, v_i)_j\}))) =$$

$$[(2U-2m+1)/2^{j+1}, (2U+2m)/2^{j+1}] \times [(2V-2m+1)/2^{j+1},$$

$$(2V+2m)/2^{j+1}], \text{ for some } (u_i, v_i)_j \in e_j(B),$$

then $2U - 2m + 2 \leq 2L \leq 2U + 2m$ and $2V - 2m + 2 \leq 2M \leq 2V + 2m$. But then

$$\frac{U - m + 1}{2^j} \leq \frac{L}{2^j} \leq \frac{U + m}{2^j}, \quad \frac{V - m + 1}{2^j} \leq \frac{M}{2^j} \leq \frac{V + m}{2^j}$$

which contradicts the choice of $(a_i, b_i)_j$. Thus,

$$\rho_{j+1}(\theta_{j,j+1}(e_j(A)), \theta_{j,j+1}(e_j(B))) = 2m$$

and case 1 is established.

Now assume the result is true when $k = n$, that is,

$$\rho_j(e_j(A), e_j(B)) = \frac{\rho_{j+n}(\theta_{j,j+n}(e_j(A)), \theta_{j,j+n}(e_j(B)))}{2^n}.$$

By case 1,

$$\rho_{j+n}(\theta_{j,j+n}(e_j(A)), \theta_{j,j+n}(e_j(B))) =$$

$$\frac{\rho_{j+n+1}(\theta_{j,j+n+1}(e_j(A)), \theta_{j,j+n+1}(e_j(B)))}{2}$$

which implies that the result it true when $k = n + 1$. Thus, the result is true for all positive k.

We can now define a distance function ρ_∞ from $A^\infty \times A^\infty$ into the nonnegative reals by

$$\rho_\infty([e_n(A)], [e_m(B)]) = \frac{\rho_n(e_n(A), \theta_{m,n}(e_m(B)))}{2^n}, \quad \text{where } m \leq n.$$

<u>Claim</u>. The function ρ_∞ is well-defined.

Suppose that $[e_n(A)] = [e_j(A')]$ and $[e_m(B)] = [e_k(B')]$.
Then there is an integer q such that n, m, j, and k are all
less than or equal to q, and $\theta_{n,q}(e_n(A)) = \theta_{j,q}(e_j(A'))$ and
$\theta_{m,q}(e_m(B)) = \theta_{k,q}(e_k(B'))$. Assume that $j \leq k$ and $m \leq n$.
By Lemma 3.9 we get that

$$\rho_\infty([e_n(A)],[e_m(B)]) = \frac{\rho_n(e_n(A),\theta_{m,n}(e_m(B)))}{2^n} =$$

$$\frac{\rho_q(\theta_{n,q}(e_n(A)),\theta_{m,q}(e_m(B)))}{2^q} =$$

$$\frac{\rho_q(\theta_{j,q}(e_j(A')),\theta_{k,q}(e_k(B')))}{2^q} =$$

$$\frac{\rho_k(\theta_{j,k}(e_j(A')),e_k(B'))}{2^k} = \rho_\infty([e_j(A')],[e_k(B')]).$$

<u>Claim</u>. The function ρ_∞ is a metric on $A^\infty \times A^\infty$.

This follows directly from Lemma 3.9 and the fact that ρ_n
is a metric for all positive integers n.

So we now have a metric space (A^∞, ρ_∞). Consider the
Cauchy completion of A , denoted by $C(A^\infty)$. Extend ρ_∞ to
$C(A^\infty)$ in the usual manner by defining

$$\hat{\rho}_\infty(\lim_{n \to \infty}[e_n(A_n)],\lim_{n \to \infty}[e_n(B_n)]) = \lim_{n \to \infty} \rho_\infty([e_n(A_n)],[e_n(B_n)]).$$

The following lemma will be used to establish a homeomorphism between $(\underline{A}, \rho_H)$ and $(C(A^\infty), \hat{\rho}_\infty)$.

<u>Lemma 3.10</u>. If $m \le n$, then

$$\rho_H(\phi_n e_n A, \phi_m e_m B) = \frac{\rho_n(e_n(A), \theta_{m,n}(e_m(B)))}{2^n}.$$

<u>Proof</u>. Because $\phi_m e_m B = \phi_n(\theta_{m,n}(e_m(B)))$, we can reduce the problem to showing that

$$\rho_H(\phi_n e_n A, \phi_n e_n B) = \frac{\rho_n(e_n(A), e_n(B))}{2^n}.$$

Let

$$e_n(A) = \cup_{j=1}^k \{(a_i, b_i)_n\},$$

$$\phi_n e_n A = \cup_{j=1}^k \{[L_j/2^n, (L_j+1)/2^n] \times [M_j/2^n, (M_j+1)/2^n]\} \text{ and}$$

$$e_n(B) = \cup_{j=1}^t \{(q_i, r_i)_n\},$$

$$\phi_n e_n B = \cup_{j=1}^t \{[Q_j/2^n, (Q_j+1)/2^n] \times [R_j/2^n, (R_j+1)/2^n]\}.$$

<u>Claim</u>. There exists an integer z such that

$$\rho_H(\phi_n e_n A, \phi_n e_n B) = z/2^n$$

Let $\rho_H(\phi_n e_n A, \phi_n e_n B) = r$ and suppose $(z-1)/2^n < r < z/2^n$ for some integer z. Then for all $j = 1, 2, \ldots, k$ and for some h an element of $\{1, 2, \ldots, t\}$,

$$\phi_n(\{(a_{i_j}, b_{i_j})_n\}) = [L_j/2^n, (L_j+1)/2^n] \times [M_j/2^n, (M_j+1)/2^n]$$

$$\subset [Q_h/2^n - r, (Q_h+1)/2^n + r] \times [R_h/2^n - r, (R_h+1)/2^n + r].$$

This implies that

$$\phi_n(\{(a_{i_j}, b_{i_j})_n\}) \subset$$

$$[(Q_h-z+1)/2^n, (Q_h+z)/2^n] \times [(R_h-z+1)/2^n, (R_h+z)/2^n].$$

So for every element x of $\phi_n e_n A$ there is an element y of $\phi_n e_n B$ so that $d(x,y) \le (z-1)/2^n$, and a symmetric argument shows the converse. Thus,

$$\rho_H(\phi_n e_n A, \phi_n e_n B) \le (z-1)/2^n < r,$$

which is a contradiction. So

$$\rho_H(\phi_n e_n A, \phi_n e_n B) = z/2^n \text{ for some integer } z.$$

By Lemma 3.2 and Lemma 3.3, this implies that

$$z - 1 \le \rho_n(e_n(A), e_n(B)) \le z.$$

Suppose that $\rho_n(e_n(A), e_n(B)) = z - 1$. Then by the one-to-one correspondence ϕ_n, this implies that for all $j = 1, 2, \ldots, k$, there is an element h of $\{1, 2, \ldots, t\}$ so that

$$[L_j/2^n, (L_j+1)/2^n] \times [M_j/2^n, (M_j+1)/2^n] \subset$$

$$[(Q_h-z+1)/2^n, (Q_h+z)/2^n] \times [(R_h-z+1)/2^n, (R_h+z)/2^n].$$

Again we get that

$$\rho_H(\phi_n e_n A, \phi_n e_n B) \leq (z-1)/2^n < z/2^n,$$

a contradiction. This implies that $\rho_n(e_n(A), e_n(B)) = z$, and the conclusion follows.

We are now able to show the connection between $(\underline{A}, \rho_H)$ and the discrete spaces (A_n, ρ_n).

<u>Theorem 3.11</u>. $(\underline{A}, \rho_H)$ is homeomorphic to $(C(A^\infty), \hat{\rho}_\infty)$.

<u>Proof</u>. Define a map $\Psi: \underline{A} \to C(A^\infty)$ by

$$\Psi(A) = \lim_{n \to \infty} [e_n(A)].$$

Note that $A = \cap_{n=1}^{\infty} \phi_n e_n A$. Suppose $\Psi(A) = \Psi(B)$. Then

$$\lim_{n \to \infty} [e_n(A)] = \lim_{n \to \infty} [e_n(B)]$$

which implies that

$$0 = \lim_{n \to \infty} \rho_\infty([e_n(A)], [e_n(B)]) = \lim_{n \to \infty} \frac{\rho_n(e_n(A), e_n(B))}{2^n} =$$

$$\rho_H(A, B)$$

by Theorem 3.4. But ρ_H is a metric, so $A = B$ and Ψ is a one-to-one map. To see that Ψ is onto, suppose that $\{[e_n(A_n)]: n \geq 1\}$ is a Cauchy sequence in $C(A^\infty)$. Then for every positive number ε, there exists an integer N such that

$\hat{\rho}_\infty([e_n(A_n)], [e_m(A_m)]) < \epsilon$ if m and n are greater than N. But

$$\hat{\rho}_\infty([e_n(A_n)], [e_m(A_m)]) = \rho_\infty([e_n(A_n)], [e_m(A_m)]) =$$

$$\frac{\rho_n(e_n(A_n), \theta_{m,n}(e_m(A_m)))}{2^n} = \rho_H(\phi_n e_n A_n, \phi_m e_m A_m) \text{ where } m \leq n.$$

Thus, $\{\phi_n e_n A_n \colon n \geq 1\}$ is a Cauchy sequence in $\underline{A}$ and $(\underline{A}, \rho_H)$ is complete [3], so there is an element A of $\underline{A}$ so that

$$\lim_{n \to \infty} \phi_n e_n A_n = A.$$

But

$$\lim_{n \to \infty} \phi_n e_n A = A,$$

so by the triangle inequality, for every positive ϵ there is an integer N so that $\rho_H(\phi_n e_n A_n, \phi_n e_n A) < \epsilon$ if $n > N$.

<u>Claim</u>. $\lim_{n \to \infty} [e_n(A_n)] = \lim_{n \to \infty} [e_n(A)]$.

Let ϵ be positive, then there is an N_ϵ such that $\rho_H(\phi_n e_n A_n, \phi_n e_n A) < \epsilon$ if $n > N_\epsilon$. Then

$$\hat{\rho}_\infty([e_n(A_n)], [e_n(A)]) = \rho_\infty([e_n(A_n)], [e_n(A)]) =$$

$$\frac{\rho_n(e_n(A_n), e_n(A))}{2^n} = \rho_H(\phi_n e_n A_n, \phi_n e_n A) < \epsilon \text{ if } n > N_\epsilon.$$

Thus,

$$\lim_{n\to\infty} [e_n(A_n)] = \lim_{n\to\infty} [e_n(A)] = \Psi(A),$$

proving that Ψ is onto.

To show that Ψ is a homeomorphism, it suffices to show that Ψ is an isometry. But this follows from Theorem 3.4, as

$$\hat{\rho}_\infty(\Psi(A),\Psi(B)) = \lim_{n\to\infty} \rho_\infty([e_n(A)],[e_n(B)]) =$$

$$\lim_{n\to\infty} \frac{\rho_n(e_n(A),e_n(B))}{2^n} = \rho_H(A,B).$$

Thus, Ψ is a homeomorphism from $(\underline{A},\rho_H)$ onto $(C(A^\infty),\hat{\rho}_\infty)$.

CHAPTER IV

APPLICATION TO DIGITIZED GREY LEVEL IMAGES

This chapter examines the possible usefulness of the metric ρ in pattern recognition problems. In particular, graded patterns (digitized grey level images) are introduced and an algorithm for computing the discrete Hausdorff distance between two such graded patterns is given.

Definition 4.1. Let N_1, N_2, and N_3 be positive integers. A graded pattern (or digitized grey level image) is an $N_1 \times N_2$ matrix with entries from $\{0,1,2,...,N_3\}$.

Such patterns are produced by high speed scanning devices, such as densitometers. These digitized images arise in areas such as land use studies, planetary observations, fingerprint analysis, X-ray diagnosis, and optical character recognition. An excellent example of the use of graded patterns is given in a study of the relationship between blood flow and brain function by Lassen et al. [10].

The discrete Hausdorff metric can be used to define a distance between graded patterns for fixed N_1, N_2, and N_3, thus giving a measure of similarity between patterns. This interset distance can then be used with a clustering algorithm for the purpose of pattern classification. The

following description of graded patterns as ordered triples of integers and the definition of neighborhoods of points is due to Bednarek and Ulam [2].

Fix integers N_1, N_2, and N_3. Let

$$X = \{(i,j,k): 1 \leq i \leq N_1, 1 \leq j \leq N_2, 1 \leq k \leq N_3\}.$$

Associate with every graded pattern $A = [a_{ij}]$ a subset $\underline{A}$ of X in the following way:

$$\underline{A} = \{(i,j,h) \in X: \text{the ij-th entry of A is k} \neq 0 \text{ and}$$

$$1 \leq h \leq k\}.$$

One can visualize $\underline{A}$ as stacking blocks on an $N_1 \times N_2$ grid, where there are k blocks stacked on (i,j) if $a_{ij} = k$, and no blocks on (i,j) if $a_{ij} = 0$.

For a point $x = (i,j,k) \in X$, define the neighborhood of x, N(x) by

$$N(x) = \{(p,q,r) \in X: |p - i| + |q - j| + r - k \leq 1$$

$$\text{and } r \geq k\}.$$

Thinking of this as blocks on an $N_1 \times N_2$ grid, this forms a partial cruciform neighborhood which consists of all those blocks immediately to the left, right, front, back, and above the block x. This is shown schematically in Figure IV-1, where x is the center cell.
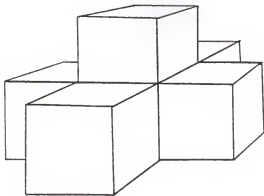
Figure IV-1

Given two graded patterns A and B, ρ is a metric on the associated sets $\underline{A}$ and $\underline{B}$ of X. An example of ρ applied to two 30 × 30 graded patterns with 15 grey levels using the above choice of neighborhoods is shown in the Appendix. The expansion of each pattern is displayed until the absorption of the other pattern is achieved.

Let $x = (i,j,k)$ be an element of $\underline{A}$, where A is a graded pattern and $N(x)$ is defined as before.

Proposition 4.2. $E^n(\{x\}) = \{(p,q,r) \in X: |p - i| + |q - j| + r - k \leq n \text{ and } r \geq k\}$.

Proof. The proof proceeds by induction on n. The case $n = 1$ is given by the definition of $N(x)$. Assume the result

is true for case n and consider $E^{n+1}(\{(i,j,k)\}) = E^{n+1}(\{x\})$
$= E(E^n(\{x\})) = \bigcup\{N((p,q,r)): (p,q,r) \in E^n(\{x\})\}$. If (p,q,r)
is an element of $E^n(\{x\})$, then

$$|p - i| + |q - j| + r - k \leq n \text{ and } r \geq k$$

by the inductive hypothesis. So if $(s,t,u) \in N((p,q,r))$,
then $|s - i| + |t - j| + u - k \leq |s - p| + |p - i| + |t - q|$
$+ |q - j| + u - r + r - k \leq n + 1$ and $u \geq r \geq k$. Thus

$$E^{n+1}(\{x\}) \subset \{(p,q,r) \in X: |p - i| + |q - j| + r - k$$

$$\leq n + 1 \text{ and } r \geq k\}.$$

Conversely, if $(p,q,r) \in X$ is such that $|p - i| + |q - j| +$
$r - k \leq n$ and $r \geq k$, then $(p,q,r) \in E^n(\{x\}) \subset E^{n+1}(\{x\})$.
If $|p - i| + |q - j| + r - k = n + 1$ and $r \geq k$, then there
is an element (s,t,u) in X such that $|p - s| + |q - t| + r -$
$u = 1$, $r \geq u$, and $|s - i| + |t - j| + u - k = n$, $u \geq k$. To
see this, suppose $k + 1 \leq r \leq k + n + 1$ and let $s = p$, $t = q$,
and $u = r - 1$. If $r = k$, then $|p - i| + |q - j| = n + 1$,
and we must consider several cases. If $p = i$ and $q = j + n$
$+ 1$, let $s = i$ and $t = j + n$; it $p = i$ and $q = j - n - 1$,
let $s = i$ and $t = j - n$. A similar argument establishes the
case where $q = j$, so assume $|p - i| > 0$ and $|q - j| > 0$.
If $p \geq i + 1$ let $s = i - 1$ and $t = q$; if $p \leq i - 1$ let $s =$
$p + 1$ and $t = q$.

So $(p,q,r) \in N((s,t,u)) \subset E(E^n(\{x\})) = E^{n+1}(\{x\})$, which implies that $E^n(\{(i,j,k)\}) = \{(p,q,r) \in X: |p-i| + |q-j| + r - k \leq n \text{ and } r \geq k\}$ for all nonnegative integers n.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two graded patterns. Associated with A and B are subsets $\underline{A}$ and $\underline{B}$ of X. The discrete Hausdorff metric defines a distance between $\underline{A}$ and $\underline{B}$:

$$\rho(\underline{A},\underline{B}) = \min \{k: \underline{A} \subset E^k(\underline{B}) \text{ and } \underline{B} \subset E^k(\underline{A})\}.$$

We now provide an algorithm for computing $\rho(\underline{A},\underline{B})$.

<u>Proposition 4.3.</u> Let

$$d_A = \underset{(i,j,k) \in \underline{A}}{\text{maximum}} \quad \underset{(p,q,r) \in \underline{B}}{\text{minimum}} \quad \{|i-p| + |q-j| + \{^{k-r \text{ if } k \geq r}_{0 \text{ otherwise}}\}$$

$$d_B = \underset{(p,q,r) \in \underline{B}}{\text{maximum}} \quad \underset{(i,j,k) \in \underline{A}}{\text{minimum}} \quad \{|i-p| + |q-j| + \{^{r-k \text{ if } r \geq k}_{0 \text{ otherwise}}\}$$

then $\rho(\underline{A},\underline{B}) = \max \{d_A, d_B\}$.

<u>Proof.</u> Let (i,j,k) be an element of $\underline{A}$, where $1 \leq k \leq a_{ij}$. For all elements (p,q,r) of $\underline{B}$, where $1 \leq r \leq b_{pq}$, compute $m_{ij} = \underset{(p,q,r) \in \underline{B}}{\text{minimum}} \quad \{|p-i| + |q-j| \text{ if } k \leq r, |p-i| + |q-j| + k - r \text{ if } k > r\}$. Then $(i,j,k) \in E^{m_{ij}}(\underline{B})$. Take the maximum over all m_{ij}, $d_A = \underset{(i,j,k) \in \underline{A}}{\text{maximum}} \quad \{m_{ij}\}$, then $\underline{A} \subset E^{d_A}(\underline{B})$. A symmetric argument shows that $\underline{B} \subset E^{d_B}(\underline{A})$, which implies that $\rho(\underline{A},\underline{B}) \leq \max \{d_A, d_B\}$. Let $m = \max \{d_A, d_B\}$ and suppose $\rho(\underline{A},\underline{B}) < m$. Then $\underline{A} \subset E^{m-1}(\underline{B})$ and $\underline{B} \subset E^{m-1}(\underline{A})$, so for all elements

(i,j,k) of $\underline{A}$ there exists an element (p,q,r) of $\underline{B}$ such that $(i,j,k) \in E^{m-1}(\{(p,q,r)\})$. This implies that $|p-i| + |q-j| + k - r \le m - 1$ and $k \ge r$. By minimizing over all (p,q,r) in $\underline{B}$ and maximizing over all (i,j,k) in $\underline{A}$, this implies that $d_A \le m - 1$. A symmetric argument shows that $d_B \le m - 1$, but this contradicts m being the maximum of d_A and d_B. Thus $\rho(\underline{A},\underline{B}) = m$.

Note that the algorithm given in Proposition 4.3 agrees with the definition of the Hausdorff metric. In actual computer computations of the distance between graded patterns, the algorithm was not used. Rather, these distances were computed directly from the definition of ρ. Iterations of the operator E for the purpose of computing ρ are shown in the Appendix.

Other metrics on graded patterns that have been used in pattern recognition are the Hamming distance H given by

$$H(A,B) = \sum_{i,j} |a_{ij} - b_{ij}|,$$

and the metric M given by

$$M(A,B) = \max_{i,j} |a_{ij} - b_{ij}|,$$

or more generally,

$$M(A,B) = \max_{i,j} \{d(a_{ij},b_{ij})\} \quad \text{where d is a metric.}$$

It is our opinion that the metric ρ is a useful measure of similarity between graded patterns, and perhaps better than other metrics now in use as it seems to be less sensitive to small perturbations or changes. Such perturbations often result from <u>noise</u> or electronic interference in real data collecting and transmitting situations. Generally, graded patterns are subjected to an image enhancement procedure before being interpreted. We shall digress slightly to describe one such procedure— a procedure that seems particularly appropriate for digitized grey level images.

CHAPTER V

A SHARPENING TRANSFORMATION FOR GRADED PATTERNS

In this chapter we discuss a nonlinear transformation introduced by Kramer and Bruchner [9] that can be used for sharpening digitized grey level images. An alternative, and in our opinion simpler, proof of the Kramer-Bruchner Theorem is given.

As discussed previously, graded patterns often become distorted or "fuzzy" and a preprocessing sharpening may be performed in order to attempt to restore the pattern to its original state. Fourier and Laplacian transformations have been used for this purpose. A simple nonlinear transformation for sharpening digitized grey level images which depends on local operations was introduced by Kramer and Bruchner [9]. Basically the transformation replaces the value of an entry of a graded pattern A by the largest or smallest value in its neighborhood. The following definition given for the sharpening S is due to Kramer and Bruchner [9].

Rather than describe the transformation in terms of graded patterns, it is best described in terms of real valued functions F on finite sets X. In the cases of interest to us though, X will be the cells of an $N_1 \times N_2$ grid, that is,

$$X = \{(i,j): 1 \le i \le N_1, 1 \le j \le N_2\},$$

and F will be the function on X which assigns to the ij-th cell the value a_{ij}, $F((i,j)) = a_{ij}$, where $A = [a_{ij}]$ is the graded pattern.

The definition of the sharpening transformation requires the notion of a <u>neighborhood system</u> for X. For every point x of X, associate with it a unique nonempty subset N(x) of X such that $x \in N(x)$, and we require that the neighborhoods satisfy a symmetry condition, that is, if $x \in N(y)$ then $y \in N(x)$ for all elements x and y of X. Notice that this definition of neighborhood is slightly more restrictive than that previously used. For every real valued function F on X, associate with it two other functions $\overline{F}$ and $\underline{F}$, the local maximum and local minimum functions respectively; that is,

$$\overline{F}(x) = \max \{F(y): y \in N(x)\} \text{ and}$$

$$\underline{F}(x) = \min \{F(y): y \in N(x)\}$$

for all elements x of X.

If X is a finite set with a neighborhood system and F is a real valued function on X, then the sharpening transformation S is defined by

$$S(F)(x) = (SF)(x) = \begin{cases} \overline{F}(x) \text{ if } \overline{F}(x) - F(x) \leq F(x) - \underline{F}(x) \\ \underline{F}(x) \text{ otherwise.} \end{cases}$$

Define $S^0F = F$, and for all positive integers n, $S^{n+1}F = S(S^nF)$ is given by

$$S^{n+1}F(x) = \begin{cases} \overline{S^nF}(x) & \text{if } \overline{S^nF}(x) - S^nF(x) \le S^nF(x) - \underline{S^nF}(x) \\ \underline{S^nF}(x) & \text{otherwise.} \end{cases}$$

Kramer and Bruchner prove the pointwise convergence of the sequence $\{S^nF\}$. We provide a direct proof based on the cardinality of the range.

A point $x \in X$ is called a <u>local maximum of F</u> if $F(x) = \overline{F}(x)$. Dually, if $F(x) = \underline{F}(x)$ we say that x is a <u>local minimum of F</u>.

It is immediate that pointwise convergence of the sequence $\{S^nF\}$ is equivalent to the assertion that there exists a positive integer N such that for each $x \in X$, x is either a local minimum or local maximum of S^NF.

<u>Theorem 5.1</u>. If X is a finite set with a neighborhood system and F is a real valued function on X, then for every element x of X, there is an integer N such that $n \ge N$ implies that $S^nF(x) = S^NF(x)$.

<u>Proof</u>. If F is constant on X then $\overline{F}(x) = \underline{F}(x) = F(x)$ for all x in X, and the result is immediate. If the cardinality of $F(X)$, $|F(X)|$, equals two, then every point of X is a local maximum or local minimum of F, and again the result follows. The proof proceeds by induction on the cardinality of $F(X)$. Assume that $|F(X)| \ge 3$, and that the result is true for all

functions F and finite sets X such that $|F(X)| \leq n$. Let $|F(X)| = n + 1$.

Let $u_0 = \max \{F(x): x \in X\}$ and $l_0 = \min \{F(x): x \in X\}$. Define $M(F) = \{x \in X: F(x) = u_0\}$ and $L(F) = \{x \in X: F(x) = l_0\}$.

Note that if $F(x) = u_0 = \overline{F}(x)$, then $S^k F(x) = \overline{S^{k-1}F}(x) = u_0$ for all positive integers k, and similarly, if $F(x) = l_0 = \underline{F}(x)$, then $S^k F(x) = \underline{S^{k-1}F}(x)$ for all positive integers k. This implies that

$$M(F) \subset M(SF) \subset M(S^2 F) \subset \ldots \subset M(S^n F) \subset \ldots \subset X, \text{ and}$$

$$L(F) \subset L(SF) \subset L(S^2 F) \subset \ldots \subset L(S^n F) \subset \ldots \subset X.$$

Because X is finite there are integers N_1 and N_2 such that

$$M(S^{N_1}F) = M(S^{N_1+k}F) \text{ and } L(S^{N_2}F) = L(S^{N_2+k}F)$$

for all nonnegative integers k. Let $N = \max \{N_1, N_2\}$, and $\underline{U} = M(S^N F)$, $\underline{L} = L(S^N F)$.

Let x be an element of $X - (\underline{U} \cup \underline{L})$.

<u>Claim</u>. Either $N(x) \cap \underline{U} = \phi$ or $N(x) \cap \underline{L} = \phi$.

Suppose that $N(x) \cap \underline{U}$ and $N(x) \cap \underline{L}$ are nonempty. Then there are points y_1 and y_2 such that $y_1 \in N(x)$ and $S^N F(y_1) = u_0$, and $y_2 \in N(x)$ and $S^N F(y_2) = l_0$. But then

$$S^{N+1}F(x) = \{ \begin{array}{l} \overline{S^N F}(x) = u_0 \text{ or} \\ \underline{S^N F}(x) = l_0 \end{array}$$

which implies that $x \in \underline{U} \cup \underline{L}$, a contradiction.

We now consider three cases.

Case 1. Suppose $N(x) \cap \underline{U} \neq \phi$. Then there is a $y \in N(x)$ such that $S^N F(y) = u_0$, and $S^N F(x) \neq u_0$ so $S^N F(x) = \underline{S^{N-1}F}(x) < u_0$. Because $\overline{S^{N+k}F}(x) = u_0$ for all $k = 0,1,2,...$ and x not in $\underline{U}$, this means that $S^{N+k}F(x) = \underline{S^{N+k-1}F}(x)$ for all nonnegative integers k. Then

$$S^N F(x) = \underline{S^{N-1}F}(x) \geq \underline{S^N F}(x) \geq \underline{S^{N+1}F}(x) \geq ... > l_0.$$

So there is an integer N_3 such that $S^{N_3}F(x) = S^{N_3+k}F(x)$ for all nonnegative integers k.

Case 2. If $N(x) \cap \underline{L}$ is nonempty then there is a $y \in N(x)$ such that $S^N F(y) = S^{N+k}F(y) = l_0$ for all $k = 0,1,2,...$. But x not in $\underline{L}$ implies that $S^{N+k}F(x) = \overline{S^{N+k-1}F}(x)$ for all nonnegative k. So

$$S^N F(x) \leq S^{N+1}F(x) \leq S^{N+2}F(x) \leq ... < u_0.$$

Thus, there is an integer N_4 such that $S^{N_4+k}F(x) = S^{N_4}F(x)$ for all nonnegative integers k.

Case 3. Suppose that both N(x) ∩ U and N(x) ∩ L are empty.
Let Y = X - (U ∪ L), F_* = F|_Y, and for all y ∈ Y, let
N_Y(y) = N(y) ∩ Y. Then |F_*(Y)| < n, so by the inductive
hypothesis there is an integer N_5 such that

$$S^{N_5}F_*(x) = S^{N_5+k}F_*(x)$$

for all nonnegative integers k. But N(x) ∩ U = φ = N(x) ∩ L
implies that N(x) ∩ Y = N(x), so SF_*(x) = SF(x). Moreover,
$S^k F_*(x) = S^k F(x)$ since $\underline{S^k F_*}(x) = \underline{S^k F}(x)$ and $\overline{S^k F_*}(x) = \overline{S^k F}(x)$
for all integers k. Thus, $S^{N_5}F(x) = S^{N_5}F_*(x) = S^{N_5+k}F_*(x) = S^{N_5+k}F(x)$ for all nonnegative integers k, and the result is
true for all x in X.

Remark. The Kramer-Bruchner claim of pointwise convergence
of {S^nF} to a function P in then established by letting
$P = S^{N_*}F$, where N_* is the maximum of the integers given by
Theorem 5.1 for the individual elements of X.

We now provide an illustration of how the sharpening
S can be applied to character recognition. Let X be an
8 × 8 matrix and let F be the function from X into
{0,1,2,...,7} which represents the graded pattern A shown in
Figure V-1 in which there are eight grey levels. We then
distorted the pattern in a random fashion by using random
numbers. The fuzzy pattern is represented by T in Figure V-1,
where T: X → {0,1,...,7} is given by T(x) = F(x) + n(x) if

$0 \leq F(x) + n(x) \leq 7$, $T(x) = F(x)$ otherwise. The noise $n(x)$ introduced was determined by first generating a random number r from $\{0,1,\ldots,9\}$ and then using the following rule:

if $r = 0$ or 9 then $n(x) = 0$

if $r = 1$ or 5 then $n(x) = 1$

if $r = 2$ or 6 then $n(x) = -1$

if $r = 3$ or 7 then $n(x) = 2$

if $r = 4$ or 8 then $n(x) = -2$.

After two applications of the sharpening transformation S, the limit $S^2 T$ was reached and found to very close to the original pattern, as shown in Figure V-1.

```
0  7  7  7  7  0  0  0        1  7  6  7  7  1  0  0
0  7  0  0  7  0  0  0        2  6  2  1  7  1  0  0
0  7  0  0  7  0  0  0        0  6  2  0  5  0  1  2
0  7  7  7  7  7  0  0        1  6  6  7  7  7  1  2
0  7  0  0  0  7  0  0        2  7  1  0  0  7  0  0
0  7  0  0  0  7  0  0        0  7  0  0  0  6  2  0
0  7  0  0  0  7  0  0        0  7  0  1  2  6  0  1
0  7  7  7  7  7  0  0        0  7  7  7  5  7  2  2
```

$$A \hspace{5cm} T$$

```
1  7  7  7  7  0  0  0        0  7  7  7  7  0  0  0
0  7  1  0  7  0  0  0        0  7  0  0  7  0  0  0
0  6  0  0  7  0  2  2        0  7  0  0  7  0  2  2
0  7  7  7  7  7  0  2        0  7  7  7  7  7  0  2
0  7  0  0  0  7  0  0        0  7  0  0  0  7  0  0
0  7  0  0  0  7  0  0        0  7  0  0  0  7  0  0
0  7  0  0  0  7  0  2        0  7  0  0  0  7  0  2
0  7  7  7  7  7  0  2        0  7  7  7  7  7  0  2
```

$$ST \hspace{5cm} S^2T$$

Figure V-1

CHAPTER VI

SOME COMPUTATIONAL EXPERIMENTS

Both the metric ρ and the transformation S depend on
the notion of neighborhoods of points of a set X. Due to
this similarity, questions arise as to the relationship (if
any) between the two. Several computer experiments were
conducted in order to study this metric and its interaction
with the sharpening of graded patterns.

Computation 1. The first set of computations involved twelve
8 × 8 matrices with random entries from $\{0,1,\ldots,7\}$. The
Kramer-Bruchner sharpened limit was computed for each of the
matrices A_i, i = 1,2,...,12; denote these limits $L_{A(i)}$. The
average distance between the matrix A_i and its limit $L_{A(i)}$
was found to be 2.4167, and the average number of iterations
to sharpen A_i was 4.3333. We then computed $\rho(A_i,A_j)$ and
$\rho(L_{A(i)},L_{A(j)})$ for all i and j, i ≠ j. The mean of $\rho(A_i,A_j)$
was 3.3333 with a standard deviation of .1667; the mean of
$\rho(L_{A(i)},L_{A(j)})$ was 2.9090 with a standard deviation of .7986.
Thus, sharpening seemed to decrease the distance between
graded patterns. However, numerous examples were found
where $\rho(L_{A(i)},L_{A(j)}) > \rho(A_i,A_j)$, thus showing that no rela-
tionship of inequality could be stated between $\rho(A,B)$ and
$\rho(L_A,L_B)$ for graded patterns A and B with sharpened limits
L_A and L_B respectively.

50

Computation 2. As a comparison, the distances between A_i and A_j, and $L_{A(i)}$ and $L_{A(j)}$, $i \neq j$, were computed using the Hamming distance; i.e., $H(A_i, A_j) = \sum_{k,l} |a_{kl}^i - a_{kl}^j|$. The average distance between A_i and A_j for $i \neq j$ was 87.8182 with a standard deviation of 11.5943; the mean for $L_{A(i)}$ and $L_{A(j)}$, $i \neq j$, was 211.7121 with a standard deviation of 21.9333. Sharpening greatly increased the Hamming distance between graded patterns, whereas sharpening created little variance using the discrete Hausdorff distance. This could be expected though, as sharpening tends to increase the number of highest and lowest grey levels in the pattern. Because the range of values was so varied for the metrics H and ρ, they were normalized and are denoted by $\overline{H}$ and $\overline{\rho}$ respectively. The average of $\overline{\rho}(A_i, A_j)$ was .1667, while the average of $\overline{H}(A_i, A_j)$ was .1960. The average of $\overline{\rho}(L_{A(i)}, L_{A(j)})$ was .1455 and the average for $\overline{H}(L_{A(i)}, L_{A(j)})$ was .4726.

In normalizing ρ it was noted that the maximum value of ρ is 20, and in general, if the size of the matrix is $N_1 \times N_2$ with entries from $\{0, 1, 2, \ldots, N_3\}$ then the maximum value of ρ is $(N_1 - 1) + (N_2 - 1) + (N_3 - 1) = N_1 + N_2 + N_3 - 3$. This value is obtained, for instance, when $A = [a_{ij}]$ where $a_{11} = 1$, $a_{ij} = 0$ otherwise, and $B = [b_{ij}]$, $b_{N_1 N_2} = N_3$, $b_{ij} = 0$ otherwise.

Computation 3. The next experiment generated one hundred 16 × 16 matrices, A_i for i = 1,2,...,100, with random entries from {0,1,2,...,9}. We then computed the sharpened limit $L_{A(i)}$ for each pattern A_i, i = 1,2,...,100, and let s(i) be the number of iterations of the transformation S needed to sharpen A_i. Next we computed $\rho(A_i, L_{A(i)})$ for i = 1,2,...,100, and let p(i) be this value. The mean of p(i), i = 1,2,...,100, was 3.2900 with a standard deviation of .4777. The mean value of s(i) for i = 1,2,...,100 was found to be 7.7900 with a standard deviation of 2.2798. A histogram of the frequency of the sharpening index for the matrices A_i, i = 1,2,...,100, is shown in Figure VI-1.

Computation 4. It was conjectured that the discrete Hausdorff metric could be used as an estimator for the parameters s(i) and p(i) associated with A_i. The estimation procedure was based on the nearest neighbor rule [4] which we now describe.

Let (X,d) be a metric space and suppose we are given n pairs (x_1, θ_1), (x_2, θ_2), ..., (x_n, θ_n) where x_i is an element of X and θ_i is a parameter associated with x_i which takes on values from {1,2,...,M}. Suppose we are given an x ε X and wish to estimate its parameter θ. We say that the element x_* of $\{x_1,...,x_n\}$ is a nearest neighbor to x if d(x,x_*) = min {d(x,x_i): i = 1,2,...,n}. If x has only one nearest neighbor x_* then we let θ = θ_* be the estimate. However, it

Frequency of values for sharpening index
for 100 16 × 16 × 10 graded patterns
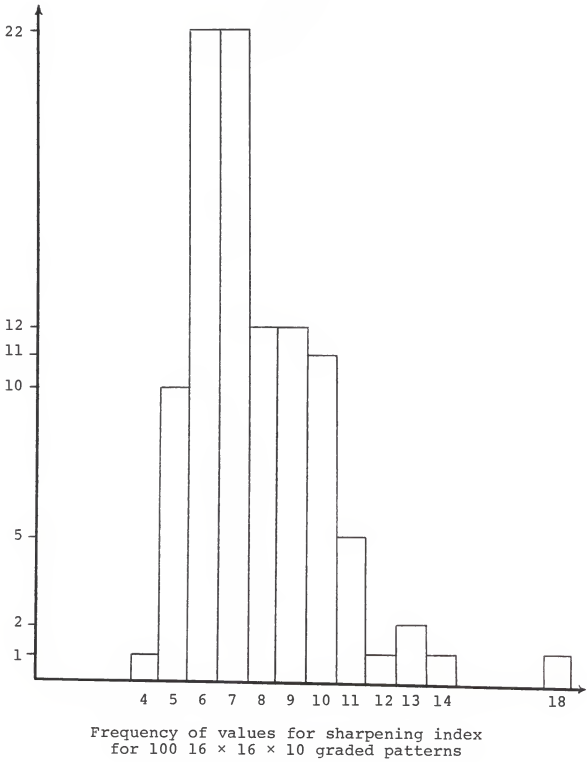
Figure VI-1

is possible for a point x to have many nearest neighbors, x_{i_1}, x_{i_2}, ..., x_{i_k}, in which case we let $\theta = \theta_{i_j}$, where the parameter θ_{i_j} occurs with the highest frequency amongst $\{\theta_{i_1}, \theta_{i_2}, ..., \theta_{i_k}\}$. In the case of ties, the choice is arbitrary.

The nearest neighbor rule was applied to the matrices A_i, taking $(A_1, s(1))$, $(A_2, s(2))$, ..., $(A_{50}, s(50))$ to be the sample set. We then computed the nearest neighbors to A_{51}, A_{52}, ..., A_{100} using three metrics: the discrete Hausdorff metric ρ, the Hamming distance H, and the maximum distance metric M given by $M(A_i, A_j) = \max_{k,l} \{|a_{kl}^i - a_{kl}^j|\}$. Of the 50 estimates made, ρ correctly estimated only 9, as did M, and H correctly estimated 12. The procedure was repeated to estimate p(i) for i = 51,52,...,100 using $(A_j, p(j))$, j = 1,2,...,50, for the sample set. This time the number of correct estimates for p(i) using ρ was 32, for M it was 29, and for H it was 28. The author's interpretation of the results was that ρ failed to be a good estimate for the parameters, particularly the sharpening index s(i), because the range of values of the metrics was too small. In particular, the values $\rho(A_i, A_j)$ for i = 51,...,100, j = 1,2,...,50 all fell between 3 and 5, whereas the possible range of values for ρ was $\{0,1,2,...,38\}$. It was felt that the range of values was small because of the nature of the matrices; the

maximum entries were uniformly distributed throughout the matrix, so that $E^k(A_i) = X$ for small k and for all $i = 1, 2, \ldots, 100$.

Computation 5. In an attempt to make the data more meaningful as a pattern recognition problem, the next computation took three prototype 16 × 16 patterns with entries from $\{0, 1, \ldots, 9\}$, representing the letters A, B, and C, and created a class of A's, class of B's, and class of C's. This was done by introducing random noise into the three prototypes, creating one sharp version and nine fuzzy versions for each type. The parameter associated with each of the patterns was either 1, 2, or 3 depending on whether it was a member of the class of A's, B's, or C's respectively.

We then took the prototype A and created 46 fuzzy versions, F_i, $i = 1, 2, \ldots, 46$, by adding random noise. The Kramer-Bruchner sharpened limits, $L_{F(i)}$, were computed for $i = 1, 2, \ldots, 46$. The mean of the sharpening index s(i) was found to be 3.4348 with a standard deviation of 1.1086. For comparison with the frequency of sharpening given in Figure VI-1, a similar histogram is given in Figure VI-2 for F_i, $i = 1, 2, \ldots, 46$. Note that the sharpening index mean dropped significantly from that given in Computation 3. This was most likely due to the fact that the entries of the patterns were no longer random, so the maximum entries were no longer uniformly distributed.
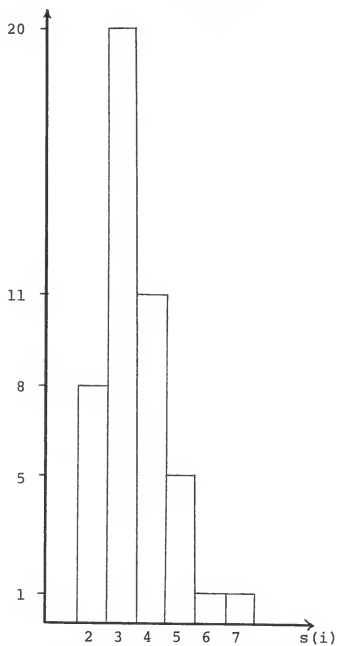
Frequency of values for sharpening index
for 46 16 × 16 × 10 graded patterns

Figure VI-2

The distances between F_i and A_j, B_j, C_j, $i = 1,\ldots,46$, $j = 1,\ldots,10$, were computed using both the discrete Hausdorff metric and the Hamming distance. Also, the distances between the sharpened versions, $L_{F(i)}$, $i = 1,2,\ldots,46$, and A_j, B_j, C_j, $j = 1,2,\ldots,10$, were computed. Based on these computations, we classified each F_i and $L_{F(i)}$ using the nearest neighbor rule. The Hamming distance correctly classified all F_i and $L_{F(i)}$, $i = 1,2,\ldots,46$, and we note that the nearest neighbor of F_i and $L_{F(i)}$ was always A_1. The mean of the nearest neighbor distance for F_i using the Hamming distance was 154.7391 with a standard deviation of 12.5423; the mean nearest neighbor distance for the sharpened version was 120.4130 with a standard deviation of 14.3319. Thus, unlike Computation 2, sharpening tended to decrease distances.

When the discrete Hausdorff metric was used to classify the fuzzy A's, F_i, $i = 1,\ldots,46$, based on nearest neighbors, 42 were correctly classified, or 91.304 percent. When the classification was made using the sharpened versions $L_{F(i)}$, 44/46 or 95.652 percent were correctly classified. The average nearest neighbor distance for F_i was 2.3478 with a standard deviation of .4815, and the average nearest neighbor distance for the sharpened versions decreased to 2.2826 with a standard deviation of .4552.

We observed that the only time the discrete Hausdorff metric misclassified an F_i or $L_{F(i)}$ was when there was one more nearest neighbor in the B class than in the A class. This was most likely because the distance between the prototypes A and B was small, $\rho(A_1, B_1) = 3$. We altered the pattern A_1 so that $\rho(A_1, B_1) = 4$ and repeated the computations. This time both ρ and H correctly classified all F_i and $L_{F(i)}$.

These computations show that the metric ρ might be useful in character recognition problems, especially if the size of the matrices was sufficiently large enough to allow significant distances between pattern types.

CHAPTER VII

POSSIBLE APPLICATIONS AND PROBLEMS

The computations presented in Chapter VI suggest that the discrete Hausdorff metric could be useful in pattern recognition problems such as character recognition. Whether it is truely a better similarity measure on graded patterns than those in existence remains to be seen. It does have some advantages which seem to make it better, at least from a geometric standpoint. That is, the choice of neighborhoods for the discrete Hausdorff metric reflects a geometrical structure on the subsets of a set X. This is not the case for other similarity measures in use, for instance, the Hamming distance. The usefulness of a geometric structure is evident in such problems as photo interpretation and analysis. These problems arise, for instance, in land use studies where the photo is taken by satellite and transmitted to earth as a digitized grey level image. Such studies are conducted regularly by private and government agencies in an attempt to study the evolution of land use, and also to locate deposits of minerals and other natural resources (such as oil).

Another instance of the use of graded patterns comes from satellite and telescopic photos of planetary observations.

For example, Strom and Strom [14] studied the evolution of disk galaxies with the aid of the interactive picture-processing system (IPPS) developed at the Kitt Peak National Observatory. Black and white photos are made with the use of a telescope in the ultraviolet to red regions of the spectrum. The various light spectrum photographs are weighted and composed to produce a digitized grey level image in which the various grey levels are color coded. The discrete Hausdorff metric provides a measure of similarity between such images.

In Chapter IV we alluded to the use of graded patterns in a study of the relationship between brain function and blood flow by Lassen et al. [10]. Specifically, they studied the changes in blood flow in areas of the human cerebral cortex in relation to specific sensory, motor, and mental activities performed by the subject. Their method of study was based on the idea that localized increased blood flow corresponds to an increase in local activity of the surrounding tissue. The study was done by injecting a small amount of a radioactive isotope, xenon 133, into the carotid artery in the neck, and measuring the arrival and subsequent wash-out of the radioactivity. The measurement was made by a gamma-ray camera consisting of 254 externally placed scintillation detectors, each detector measuring approximately one square centimeter of brain surface. The data were processed by computer and displayed on a color-television

screen, with different flow levels being assigned different colors or hues. These scientists were able to show a correlation between specific mental stimulation and actual activities being performed by the subject.

Because blood flow increases and decreases are localized, it is conjectured that the discrete Hausdorff metric could be of use in analyzing such data. Land use studies also share this attribute of localization which makes them particularly suited for a similarity measure, such as the discrete Hausdorff metric, which is dependent on a neighborhood system.

Other possible applications of the metric ρ arise in taxonomy, in particular, as a paleontological dissimilarity measure as discussed by Bednarek and Smith in [1]. Many of the taxonomic distances in existence have very complex algorithms, and often they are not true metrics. The discrete Hausdorff metric is simply stated and generally easily computed.

Because of the generality of the metric ρ, it is conjectured that a number of metrics introduced earlier as evolutionary distances will be specific cases of ρ. Sellers [12-13] provides an algorithm for an evolutionary distance that was introduced by Ulam [15] and discussed further by Waterman, Smith, and Beyer [16]. This distance on finite

sequences is metric and is used to measure the degree of evolutionary divergence between homologous proteins or nucleic acid sequences. Basically, it compares two finite sequences A and B, not necessarily the same length, and finds the common subsequences. It can be interpreted as the smallest number of weighted changes necessary to bring the two sequences into coincidence.

Several questions concerning the discrete Hausdorff metric remain open. Due to the similar dependence on a neighborhood system, the question arises as to what relationship (if any) exists between the sharpening transformation S and the metric ρ. Specifically, what conditions will insure a relationship between $\rho(A,B)$ and $\rho(L_A, L_B)$, where A and B are graded patterns and L_A and L_B are their sharpened limits respectively? We have noted instances where $\rho(A,B) > \rho(L_A, L_B)$ and $\rho(A,B) < \rho(L_A, L_B)$, showing that some additional assumptions are necessary in order to predict a relationship. Also, what relationship exists between the sharpening index and the bounds N_1, N_2, and N_3 on the graded pattern A?

Another area of investigation concerns metric semigroups. A metric semigroup is a triple $(S, \circ, d)$ where $(S, \circ)$ is a semigroup, (S,d) is a metric space, and $\circ$ is continuous with respect to the topology induced by d. More generally, a topological semigroup is a topological space G that is also

a semigroup with operation ∘ such that the mapping $(x,y) \rightarrow x \circ y$ is continuous. If (X,d) is a compact metric space, then $(2^X, \cup, \rho_H)$ is a metric semigroup, as the union operation is continuous with respect to the Hausdorff metric. Furthermore, in this space $\rho(A \cup B, C \cup D) \leq \max \{\rho(A,C), \rho(A,D), \rho(B,C), \rho(B,D)\}$. Is this the only metric semigroup having this property?

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APPENDIX

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GRADED PATTERN A
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 1 4 1 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 1 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 0 0 0 0 0 0 0 0 0 1 4 1 0 0 0 0 0 0 0 0 0 0
0 0 0 0 1 0 1 2 1 2 0 0 0 0 0 0 0 0 0 1 4 1 3 0 0 0 0 0 0 0 0 0
0 0 0 0 9 9 9 9 9 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1 3 1 3 1 2 1 0 9 0 0 0 0 0 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1 3 1 3 1 0 9 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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E  1(A)
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 1 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 9 1 0 1 1 1 2 1 3 1 4 1 4 1 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 1 4 1 4 1 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 1 0 1 1 1 2 1 3 1 4 1 4 0 0 0 0 0 0 0 1 4 1 0 0 0 0 0 0
0 0 0 1 0 1 2 1 2 1 3 1 3 0 0 0 0 0 0 0 0 0 1 4 1 4 1 0 0 0 0 0
0 0 0 9 1 2 1 2 1 3 1 3 0 0 0 0 0 0 0 0 0 1 4 1 4 1 4 1 3 1 2 1 0 9 0 0 0 0
0 0 0 9 1 9 1 2 1 2 1 0 1 3 9 0 0 0 0 0 0 0 1 4 1 4 1 4 1 4 1 3 1 2 1 0 9 0 0
0 0 0 8 9 9 9 9 9 9 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1 4 1 4 1 3 1 0 9 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 4 1 4 1 3 1 3 1 0 9 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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E   2(A)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 10 1 11 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 9 10 11 11 11 11 2 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 12 12 13 13 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 13 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 13 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 13 14 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 13 14 14 14 14 14 14 0 0 0 0 0 0 0 0 0 14 10 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 11 12 13 14 14 14 14 14 0 0 0 0 0 0 0 14 14 14 10 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9 10 12 13 14 14 14 0 0 0 0 0 0 14 14 14 14 14 14 12 10 9 0 0 0 0 0 0 0 0 0 0 0 0 0
9 12 13 14 13 13 9 0 0 0 0 0 0 14 14 14 14 14 14 13 12 10 9 0 0 0 0 0 0 0 0 0 0 0 0
9 10 12 13 13 12 11 0 9 0 0 0 0 14 14 14 14 14 14 13 12 10 9 0 0 0 0 0 0 0 0 0 0 0 0
8 9 10 12 12 1 0 10 9 0 0 0 0 0 14 14 14 14 14 14 13 10 9 0 0 0 0 0 0 0 0 0 0 0 0 0
0 8 9 9 9 9 9 0 0 0 0 0 0 0 0 14 14 14 14 14 13 10 9 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 14 14 13 13 10 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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E   3(A)
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 C 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
0 0 910111211 2 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C 0 0
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10111213141414141 4 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
10111213141414141414 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
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10111213141414141414141414 0 0 0 0141414141410 0 0 0 0 0 0 0
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10121314141414141110 9 0 0 0 0141414141414141414141 41413121 0 9 C 0
910121314141413131110 9 0 0 0 0141414141414141414141 4141 4131 0 9 0
8 910121313121110 9 0 0 0 0 0141414141414141 41414131 0 9 C C
0 8 910121210 C 0 9 0 0 0 0 0 0141414141414141 41210 9 0 0 0
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0 0 0 8 8 0 0 0 0 0 0 0 0 0 0 014141313 10 9 0 0 0 0 0
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0 0 0 0 C 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
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Ξ   5(A)
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0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 C 0 0 0 0 0 0 0 0 0 0 0
0 0 910 11 10 9 0 0 0 0 0 0 0 0 0 0 0 C C 0 0 0 0 0 0 0 0 0 0
0 910111211 2 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
910111213121313 9 0 0 0 0 0 0 0 0 0 C 0 0 0 0 0 0 0 0 0 0 C
10111213141314141 4 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
11121314141414141414 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C C
11121314141414141414 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
11121314141414141414141 4 0 0 0 0 0 0 C 0 0 0 0 0 0 0 0 0 0
11121314141414141414141 4 0 0 0 0 0 01410 0 0 0 0 0 0 0 0 0
11121314141414141414141414 0 0 0141414141410 0 0 0 0 C 0 0
11121314141414141414141414 0 0141414141414121 0 9 0 0 0 0
11121314141414141414141414 0 0141414141414141414121 0 9 0 0 0
12131414141414141414141 4 9 0 0141414141414141414141 4121210 9 C 0
13141414141414141414141 4 9 0 0141414141414141414141 4131210 9 0
12131414141414141411110 9 914141414141414141414141414131 21C 9
10121314141413131311110 9 0 0141414141414141414141 41414131 0 9
910121313131211110 9 0 0 0141414141414141414141 41413131 0 9 C C
8 910121312111 0 9 0 0 0 0141414141414141 414131 0 9 0 0
0 8 910121210 9 0 0 0 0 0 0141414141414141 41 0 9 0 0 0 0
0 0 8 9 9 9 9 0 0 0 0 0 0 0141414141 41C 9 0 0 0
0 0 8 8 0 0 0 0 0 0 0 0 0 0 01414131310 9 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C C
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   6(A)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 910 1110 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 910 1011 1211 12 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
910 11 1213 12 1313 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1011 1213 1413 1414 14 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1112 1314 1414 1414 1414 14 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 14 0 0 0 0 0 01410 0 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 14 14 0 0 014 14 1410 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 14 14 14 0 014 14 14 14 1210 9 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9 0 0 0 0
1213 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9 0 0 0
1314 14 14 14 14 14 14 14 14 14 914 14 14 14 14 14 14 14 14 14 1321 0 9 0 0
1414 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1321 0 9 0
1314 14 14 14 14 14 14 14 14 14 1114 14 14 14 14 14 14 14 14 14 14 1321 0
1213 14 14 14 14 14 14 14 14 1110 914 14 14 14 14 14 14 14 14 14 14 1310 0
1012 1314 14 14 14 14 14 1110 9 0 014 14 14 14 14 14 14 14 14 14 1310 9
910 1213 1413 1313 1110 9 0 0 0 014 14 14 14 14 14 14 14 1310 9 0
8 910 1213 13 1211 10 9 0 0 0 0 0 0 14 14 14 14 14 14 14 1310 9 0 0
0 8 910 12 1210 9 0 0 0 0 0 0 014 14 14 14 14 14 1310 9 0 0 0
0 0 8 9 9 9 9 9 0 0 0 0 0 0 0 0 014 14 14 14 131 0 9 0 0 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 014 14 131 310 9 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   7(A)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 910 1110 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 910 11 1211 12 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
910 11 1213 1213 1313 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1011 1213 1413 1414 14 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1112 1314 1414 1414 1414 14 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1213 14 14 14 14 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1314 14 14 14 14 14 14 14 14 14 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1314 14 14 14 14 14 14 14 14 14 14 0 0 0 0 0 01410 0 0 0 0 0 0 0 0
1314 14 14 14 14 14 14 14 14 14 14 14 0 0 014 14 1410 0 0 0 0 0 0 0 0
1314 14 14 14 14 14 14 14 14 14 14 14 14 0 014 14 14 14 1210 9 0 0 0 0
1314 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9 0 0 0
1314 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9 0 0
1314 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9 0
1414 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1210 9
1414 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1314 1 0
1414 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 14 1412 12
1314 14 14 14 14 14 14 14 1110 914 14 14 14 14 14 14 14 14 14 14 1413
1213 14 14 14 14 14 14 14 1110 9 0 014 14 14 14 14 14 14 14 14 1310
1012 1314 14 14 14 14 1110 9 0 014 14 14 14 14 14 14 14 14 1310 9
910 1213 1314 1413 1313 1110 9 0 0 0 014 14 14 14 14 14 14 14 1310 9 0
8 910 1213 13 1211 10 9 0 0 0 0 0 0 14 14 14 14 14 14 14 1310 9 0 0
0 8 910 12 1210 9 0 0 0 0 0 0 014 14 14 14 14 14 1310 9 0 0 0
0 0 8 9 9 9 9 9 0 0 0 0 0 0 0 0 014 14 14 14 131 0 9 0 0 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 0 14 14 131 310 9 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   8(A)
C 0 0 0 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 910111 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
C 910111211112 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
910111213121313 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1011121314131414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
1112131414141414141 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
12131414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1314141414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
141414141414141414141414 0 0 0 0 0 01410 0 0 0 0 0 0 0 0
14141414141414141414141414 0 0 0141414110 0 0 0 0 0 0 0 0
141414141414141414141414141414 0 0141414141210 9 0 0 0 0 0
1414141414141414141414141414141414141414141141210 9 0 0 0 0
14141414141414141414141414141414141414141414141141210 9 0 0
141414141414141414141414141414141414141414141414141141210 9 0 0
1414141414141414141414141414141414141414141414141414141210 9
1414141414141414141414141414141414141414141414141414141413
14141414141414141414141414141414141414141414141414141414143
14141414141414141414141414141114141414141414141414141414141414
1314141414141414141414111141414141414141414141414141414141413
1213141414131311141110 9141414141414141414141414141414141310
1012131414141414141110 9 0 014141414141414141414141414141310 9
910112131413131110 9 0 0 0 01414141414141414141414141310 9 0 0
8 910121313121113 9 0 0 0 0 0 014141414141414141310 9 0 0
0 8 910121210110 9 0 0 0 0 0 0 01414141414141310 9 0 0 0
0 0 8 9 9 9 9 0 0 0 0 0 0 0 0 0 01414141414131310 9 0 0 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 0 01414131310 9 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C
```

```
E   9(A)
0 0 0 910 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 910111 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
C 910111211112 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
910111213121313 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1011121314131414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1112131414141414141 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
12131414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1314141414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
141414141414141414141414 0 0 0 0 0 01410 0 0 0 0 0 0 0 0
14141414141414141414141414 0 0 0141414110 0 0 0 0 0 0 0 0
141414141414141414141414141414 0 01414141421210 9 0 0 0 0 0
14141414141414141414141414141414141414141414141141210 9 0 0 0 0
141414141414141414141414141414141414141414141414141141210 9 0 0
1414141414141414141414141414141414141414141414141414141141210 9 0
1414141414141414141414141414141414141414141414141414141141210 9 0
14141414141414141414141414141414141414141414141414141414141210 9
14141414141414141414141414141414141414141414141414141414141414
14141414141414141414141414141414141414141414141414141414141414
14141414141414141414141414141414141414141414141414141414141414
14141414141414141414141414141414141414141414141414141414141414
1314141414141414141414111141414141414141414141414141414141413
1213141414141414141414141110 9141414141414141414141414141310
1012131414141414141110 9 0 014141414141414141414141414141310 9
910112131413131110 9 0 0 0 01414141414141414141414141310 9 0 0
8 910121313121113 9 0 0 0 0 0 014141414141414141310 9 0 0
0 8 910121210110 9 0 0 0 0 0 0 01414141414141310 9 0 0 0
0 0 8 9 9 9 9 0 0 0 0 0 0 0 0 0 01414141414131310 9 0 0 0 0
0 0 0 8 8 8 0 0 0 0 0 0 0 0 0 0 0 0 01414131310 9 0 0 0 0
```

```
E 10(A)
  0  0 9101110 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
  C 91011121112 9 0 0 0 0 0 0 0 0 0 0 0 C 0 0 0 0 0 0 0 0 0 0
  910111213121313 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 1011121314131414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C 0
 111213141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 12131414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
 1314141414141414141414 9 0 0 0 0 0 0 0 0 0 0 0 0 0 0 C 0
 141414141414141414141414 9 0 0 0 0 01410 0 0 0 0 0 0 0 C 0
 14141414141414141414141414 9 0 0 01414141410 0 0 0 0 0 0 0
 141414141414141414141414141414 0 0141414141421210 S 0 0 0 C C
 141414141414141414141414141414141414141414141421210 9 0 C 0
 1414141414141414141414141414141414141414141414141210 9 0 C 0
 14141414141414141414141414141414141414141414141414141210 9 0
 1414141414141414141414141414141414141414141414141414141210 9 0
 14141414141414141414141414141414141414141414141414141414121C S
 1414141414141414141414141414141414141414141414141414141421C
 14141414141414141414141414141414141414141414141414141414412
 1414141414141414141414141414141414141414141414141414141414
 14141414141414141414141414141414141414141414141414141414141414
 14141414141414141414141414141414141414141414141414141414141414
 14141414141414141414141414141414141414141414141414141414141414
 14141414141414141414141414141414141414141414141414141414141414
 13141414141414141414141414411141414141414141414141414141414413
 121314141414141414141411110 91414141414141414141414141414141310
 1012131414141414141411110 9 0 014141414141414141414141414141310 S
 9101213141413131110 9 0 0 0 014141414141414141414141414141310 9 0
 B 910121313121110 S 0 0 0 0 0 014141414141414141414141310 9 C 0
 0 8 91012121010 9 0 0 0 0 0 0 0 014141414141414141310 9 0 0 0
 0 0 8 9 9 9 9 9 9 0 0 0 0 0 0 0 0 014141414141310 9 0 0 0 0
```

GRADED PATTERN B IS A SUBSET OF E 10(A)

```
E   2(B)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 01011121110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 010111121312112110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0101112131413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 010111213141414141414131211110 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 011112131414141414141414131211 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 010111121314141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 010111213141312111110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 01011121312111110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0101011110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   3(B)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0101011110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 01011121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121314131211110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141312111110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141414141414141413121110 0 0 0 0 0 0
0 0 0 0 0 0 0 011112131414141414141414141413121110 0 0 0 0 0 0
0 0 0 0 0 0 0 010111213141414141414141413121110 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 010111213141312111110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 01011121312111110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0101011110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E  4(B)
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  010  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0101110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  01011121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  010111213121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0101112131413121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  010111213141414141413121110  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  01011121314141414141414141413121110  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  011121314141414141414141414141312110  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  010111213141414141414141414141413121110  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0101112131414141414141414141413121110  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  010111213141414141413121110  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  01011121314131211C  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  01011121312110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0101112110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0101110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0010  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
```

```
E  5(B)
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  010  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0101110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  01011121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  010111213121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0101112131413121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  010111213141414141413121110  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  01011121314141414141414141413121110  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  010111213141414141414141414141413121110  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  011121314141414141414141414141413121110  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0101112131414141414141414141413121110  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  010111213141414141414141413121110  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0101112131414141413121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  010111213141413121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0101112131413121110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  010111213131211C  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0101112131211110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0101112111110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0101110  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  010  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
0  0  0   0   0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0  0
```

E   6(B)

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 1 2 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 2 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

E   7(B)

```
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 2 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 1 1 1 2 1 3 1 4 1 4 1 3 1 2 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 4 1 4 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 3 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 2 1 1 1 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   B(B)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0101121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121312111 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 01011121314141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0101112131414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 01011121314141414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 010111213141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 010111213141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0 0
0 0101112131414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0
010111213141414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0
011121314141414141414141414141414141414141413121 1 0 0 0 0 0 0 0 0 0 0
010111213141414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0
0 010111213141414141414141414141414141414143121110 0 0 0 0 0 0 0 0 0 0
0 0 010111213141414141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 01011121314141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0101112131414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0101112131413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0101112131413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0101112131211C 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 01011121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E   S(B)
0 0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0101011121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0101011121312111 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 01011121314131211 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 010111213141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0101112131414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 01011121314141414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 010111213141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 010111213141414141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0 0
0 0101112131414141414141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0
010111213141414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0 0
01011121314141414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0
011121314141414141414141414141414141414141414141413121 1 0 0 0 0 0 0 0
010111213141414141414141414141414141414141414143121110 0 0 0 0 0 0 0 0
0 0101112131414141414141414141414141414141413121110 0 0 0 0 0 0 0 0 0
0 0 010111213141414141414141414141414143121110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 01011121314141414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0101112131414141414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0101112131413121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121314131211 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0101112131211C 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0101011121110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 010 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
```

E 10(B)
```
0 0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101011121110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101112131211 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 010111213143121110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 01011121314141413121110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0101011121314141414141413121110 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 01011121314141414141414141413121110 0 0 0 0 0 0 0
0 0 0 0 0 0 010111213141414141414141414141413121110 0 0 0 0 0 0
0 0 0 0 0 0101112131414141414141414141414141413121110 0 0 0 0 0
0 0 0 010111213141414141414141414141414141413121110 0 0 0 0
0 0 010111213141414141414141414141414141414141413121110 0 0 0
0101011121314141414141414141414141414141414141413121110 0 0
10111213141414141414141414141414141414141414141413121110
11121314141414141414141414141414141414141414141413121 0
101112131414141414141414141414141414141414141413121110 0
0101112131414141414141414141414141414141413121110 0 0 0
0 0 01011121314141414141414141414141413121110 0 0 0 0
0 0 0 01011121314141414141414141413121110 0 0 0 0 0
0 0 0 0 010111213141414141414141413121110 0 0 0 0 0 0
0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0
0 0 0 0 0 0 01011121314141413121110 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 010111213141413121110 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0101112131413121110 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121311110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 010111110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 010110 0 0 0 0 0 0 0 0 0 0 0 0
```

E 11(B)
```
0 0 0 0 0 0 0 0 0 0 0 01011121110 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 010111213121110 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121314131211 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 010111213141413121110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 01011121314141413121110 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 01011121314141414141413121110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0101112131414141414141413121110 0 0 0 0 0 0 0 0 0
0 0 0 0101112131414141414141414141413121110 0 0 0 0 0 0 0 0
0 0 0101112131414141414141414141414141413121110 0 0 0 0 0
0 0 0101011121314141414141414141414141414141413121110 0 0 0
010111213141414141414141414141414141414141413121110 0 0
1011121314141414141414141414141414141414141413121110 0
11121314141414141414141414141414141414141414141413121 1
1213141414141414141414141414141414141414141414141413121110
1112131414141414141414141414141414141414141414141413121110
101112131414141414141414141414141414141414141413121110 0
0101112131414141414141414141414141414141413121110 0 0
0 0 01011121314141414141414141414141413121110 0 0 0
0 0 0 0101112131414141414141414141413121110 0 0 0 0
0 0 0 010111213141414141414141413121110 0 0 0 0 0
0 0 0 0 0101112131414141414141413121110 0 0 0 0 0 0
0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0
0 0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0
0 0 0 0 0 0 0 010111213143121110 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 01011121314131211 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 01011121311110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0101011121110 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0101110 0 0 0 0 0 0 0 0 0 0 0 0
```

```
E 12(B)
 0 0 0 0 0 0 0 0 0 0 010111213121110 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 010111213141312111 0 0 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0101112131414141413121110 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 010111213141414141414141413121110 0 0 0 0 0 0 0
 0 0 0 0 0 0 010111213141414141414141414141413121110 0 0 0 0 0
 0 0 0 0 0 010111213141414141414141414141414141413121110 0 0 0 0
 0 0 0 0 010111213141414141414141414141414141414141413121110 0 0 0
 0 0 0 010111213141414141414141414141414141414141414141413121110 0 0
 0 0 010111213141414141414141414141414141414141414141414141413121110 0 0
 0 010111213141414141414141414141414141414141414141414141414141413121110 0 0
 10111213141414141414141414141414141414141414141414141414141414141413121110 0
 111213141414141414141414141414141414141414141414141414141414141414141413121110
 121314141414141414141414141414141414141414141414141414141414141414141414131211
 131414141414141414141414141414141414141414141414141414141414141414141414141312
 121314141414141414141414141414141414141414141414141414141414141414141414131211
 111213141414141414141414141414141414141414141414141414141414141414141413121110
 10111213141414141414141414141414141414141414141414141414141414141413121110 0 0
 0 010111213141414141414141414141414141414141414141414141413121110 0 0
 0 0 010111213141414141414141414141414141414141414141413121110 0 0 0
 0 0 0 010111213141414141414141414141414141414141413121110 0 0 0 0
 0 0 0 0 010111213141414141414141414141414141413121110 0 0 0 0 0
 0 0 0 0 0 010111213141414141414141414141413121110 0 0 0 0 0 0
 0 0 0 0 0 0 010111213141414141414141413121110 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 010111213141414141413121110 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 010111213141413121110 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 010111213141312111 0 0 0 0 0 0 0 0 0 0
 0 0 0 0 0 0 0 0 0 0 010111213121110 0 0 0 0 0 0 0 0 0 0 0
```

GRADED PATTERN A IS A SUBSET OF E 12(B)

DISTANCE BETWEEN A AND B IS 12

BIOGRAPHICAL SKETCH

Carolyn Jean Roche Johnson was born on March 2, 1954, in Boston, Massachusetts, to William Paul and Jean Leah Roche. In 1957, the Roche family moved to Florida and have remained since. Carolyn studied mathematics at the University of Florida, receiving her Bachelor of Arts degree with high honors March 1976, and her Master of Science degree in December 1977. Throughout her graduate studies she has taught mathematics at the University of Florida. After graduating she will be a Member of Technical Staff at Bell Laboratories in Holmdel, New Jersey.

Carolyn is married to Karl Bruce Johnson, a 1976 graduate of the University of Florida. She is a member of Phi Kappa Phi Honor Society, the American Mathematical Society, and the Mathematical Association of America.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

A. R. Bednarek, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

B. B. Baird
Assistant Professor of
 Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

M. P. Hale
Associate Professor of
 Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

S. Y. Su
Professor of Electrical
 Engineering and Computer
 and Information Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

S. M. Ulam
Graduate Research Professor
 of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate Council, and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August 1980

Dean, Graduate School